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Michel Grabisch
Fen Li

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Anti-conformism in the threshold model of collective behavior*

Michel GRABISCH[1 2] and Fen LI[1]

[1] *University of Paris 1 Panthéon-Sorbonne, Paris, France*
[2] *Paris School of Economics, Paris, France*

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Abstract

We provide a first study of the threshold model, where both conformist and anti-conformist agents coexist. The paper is in the line of a previous work by the first author (Grabisch et al., 2018), whose results will be used at some point in the present paper. Our study bears essentially in answering the following question: *Given a society of agents with a certain topology of the network linking these agents, given a mechanism of influence for each agent, how the behavior/opinion of the agents will evolve with time, and in particular can it be expected that it converges to some stable situation, and in this case, which one?* Also, we are interested by the existence of cascade effects, as this may constitute a undesirable phenomenon in collective behavior. We divide our study into two parts. In the first one, we basically study the threshold model supposing a fixed complete network, where every one is connected to every one, like in the work of Granovetter (1978). We study the case of a uniform distribution of the threshold, of a Gaussian distribution, and finally give a result for arbitrary distributions, supposing there is one type of anti-conformist. In a second part, the graph is no more complete and we suppose that the neighborhood of an agent is random, drawn at each time step from a distribution. We distinguish the case where the degree (number of links) of an agent is fixed, and where there is an arbitrary degree distribution.

Keywords: threshold model; anti-conformism; absorbing class; opinion dynamics

JEL Classification: C7, D7, D85

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1 Introduction

Human behavior is governed by many aspects, related to social context, culture, law and other factors. Most of these aspects tend to indicate that our behavior is heavily influenced by the behavior of the other people with whom we are in contact, either directly or indirectly by means of communication devices, information media, etc. Behavior refers here to any kind of action, decision to be taken, or opinion to be held on a given topic. As our environment is constantly changing, behavior and opinion of people, including us, are evolving with time, which makes central the following question: *Given a society of agents with a certain topology of the network linking these agents, given a mechanism of influence for each agent, how the behavior/opinion of the agents will evolve with time, and in particular can it be expected that it converges to some stable situation, and in this case, which one?*

Evidently the question has been studied by sociologists and psychologists, and a number of pioneering models of "opinion dynamics" have been proposed by them, e.g., Granovetter (1978), Abelson (1964), French Jr (1956), Friedkin and Johnsen (1990), Taylor (1968), but it has also attracted the attention of many physicists, assimilating agents to particles (see a survey in Castellano et al. (2009)), economists (see, e.g., the monograph of Jackson (2008), and the survey by Acemoglu and Ozdaglar (2011)), computer scientists and probabilists (by analogy with (probabilistic) cellular automata, see, e.g., Gravner and Griffeath (1998) and the survey by Mossel and Tamuz (2017)), etc.

One of the simplest model of behavior/opinion dynamics when the opinion or behavior is binary (yes/no, active/inactive, action 1 or 0, etc.) is the threshold model, also called the majority rule model (Galam, 2002), proposed by Granovetter (1978), Schelling (1978), among others. This models simply says that one takes action 1 if sufficiently enough people in his neighborhood takes action 1. The simplicity of the model allows for a deep analysis (see the surveys by Mossel and Tamuz (2017) and Castellano et al. (2009)), and one remarkable result already observed in the pioneering work of Granovetter (1978) was that a cascade effect occurs, supposing that the population of agents starts from an initial state where nobody is active, and that the distribution of the threshold value is uniform over the population. Then, after a finite number of steps, all agents become active. Interestingly, the latter study was done in the context of a mob, where the available actions were "to riot" (action 1) or to be inactive (action 0). Then, agents with threshold 0 were called "instigators" as they start to riot alone, and this indeed forms the seed of the cascade effect, ending in a mob rioting. This topic has been very much studied, as demonstrated by a recent monograph on mob control (Breer et al., 2017), written by researchers in control theory.

So far, all these models make the basic assumption that agents tend to follow the trend (they are conformist) and that nobody will have a kind of opposite behavior (anti-conformism), choosing action 0 if too many people take action 1. Although the literature on opinion dynamics is vast, to the best of our knowledge, very few studies consider that agents may be anti-conformists, and even less consider that both types of agents may coexist. In game theory, such kind of opposite behavior has been studied however, in what is called anti-coordination games, see, e.g., Bramoullé et al. (2004); López-Pintado (2009), congestion games (Rosenthal, 1973), and fashion games (Cao et al., 2013). We are not aware, however, of such studies with the threshold model.

Our aim is to fill this gap, and to provide a first study of the threshold model, where both conformist and anti-conformist agents coexist. The paper is in the line of a previous work by the first author (Grabisch et al., 2018), whose results will be used at some point in the present paper. Our study bears essentially in answering the main question raised in the first paragraph, that is, on the convergence of the process, analyzing if absorbing states exist (stable state of the society) or if a cycle occurs, or even more chaotic situations. Also, we are interested by the existence of cascade effects, as this may constitute a undesirable phenomenon in collective behavior. We divide our study into two parts. In the first one, we basically study the threshold model supposing a fixed complete network, where every one is connected to every one, like in the work of Granovetter (1978) (Section 2). We study the case of a uniform distribution of the threshold, of a Gaussian distribution, and finally give a result for arbitrary distributions, supposing there is one type of anti-conformist. In a second part (Section 3), the graph is no more complete and we suppose that the neighborhood of an agent is random, drawn at each time step from a distribution. We distinguish the case where the degree (number of links) of an agent is fixed, and where there is an arbitrary degree distribution. Most of the proofs can be found in the Appendix.

2 The deterministic threshold model with anti-conformists

2.1 The model

Let $N = \{1, \dots, n\}$ be the society of agents. We suppose the existence of an underlying (exogenous) network $G = (N, E)$ whose nodes are the agents and E is the set of (undirected) edges or links. Each agent i has a set of neighbors $\Gamma_i = \{j \in N : \{i, j\} \in E\}$, and $|\Gamma_i| =: d_i$ is the degree of agent i. We consider that $i \in \Gamma_i$ for every agent i.

Two actions (or opinions, states) are available to each agent at every stage: 1 (agree, adopt, join, be active, etc.) and 0 (disagree, refuse, disjoin, be inactive, etc.). The action taken by agent i at stage t is denoted by $a_i(t)$. For short, we will often use the term "active" for agents taking action 1, and "inactive" for agents taking action 0.

In the classical *threshold model* introduced by Granovetter Granovetter (1978), and Schelling Schelling (1978) among others, agent i will take action 1 at next stage if the proportion of his neighbors taking action 1 exceeds some threshold $\mu_i \in [0, 1]$, otherwise action 0 is taken:

$$a_i(t+1) = \begin{cases} 1, & \text{if } \frac{1}{|\Gamma_i|}\sum_{j\in\Gamma_i} a_j(t) \geq \mu_i \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Note that unlike some threshold models, e.g., in Watts Watts (2002), an agent having adopted action 1 may return to action 0, because not enough neighbors take action 1.[1]

Such behavior exhibits a tendency to follow the trend, and we call this type of agent a *conformist*. The tendency to do the opposite of the trend is called *anti-conformism*, and can be modelled as follows:

$$a_i(t+1) = \begin{cases} 0, & \text{if } \frac{1}{|\Gamma_i|}\sum_{j\in\Gamma_i} a_j(t) \geq \mu_i \\ 1, & \text{otherwise.} \end{cases} \tag{2}$$

[1] When adoption of action 1 stays for ever, one speaks of "switch".

When too many people take action 1, then an anti-conformist agent takes action 0, and vice-versa. In the rest of the paper, we denote by N_a the set of anti-conformist agents, and by $N_c := N \setminus N_a$ the set of conformist agents.

Observe that thresholds 0 and 1 play a particular role. For a conformist agent (respectively, an anti-conformist agent), a threshold equal to 0 means that he takes always action 1 (respectively, 0), while a threshold strictly greater than 1 implies to always take action 0 (respectively, 1). We call these agents *constant 0-player* and *constant 1-player*.

Our aim is to study the evolution of the dynamics of actions taken by the agents. To this aim, we define the *state* of the society at stage t, as the set S (or $S(t)$) of agents taking action 1 at stage t. Depending which one is more convenient, a state is either denoted as a set $S \subseteq N$ or as its characteristic vector $\mathbb{1}_S$ in $\{0,1\}^N$. The process is deterministic and Markovian, i.e., transitions from S to T (denoted by $S \to T$) are with probability 1 and do not depend on states before S.

We are interested in finding absorbing states, i.e., such that $S(t) = S(t+1)$ for some value of t, and cycles, i.e., sequences of transitions $S_1 \to S_2 \to \cdots \to S_k$ where $S_k = S_1$.

2.2 A general result on cycles

There is a well-known result on the threshold model saying that the state converges to either a fixed state (absorbing) or a cycle of length 2. The most general form of this result is provided by Goles and Olivos Goles and Olivos (1980), where the process has the form

$$a_i(t+1) = \begin{cases} 1, & \text{if } \sum_{j\in N} \alpha_{ij} a_j(t) \geq \theta_i \\ 0, & \text{otherwise,} \end{cases}$$

with $\alpha_{ij} = \alpha_{ji} \in \mathbb{R}$, $\theta_i \in \mathbb{R}$ for all i, j. Then there exists $t \in \mathbb{N}$ such that $a_i(t+2) = a_i(t)$.

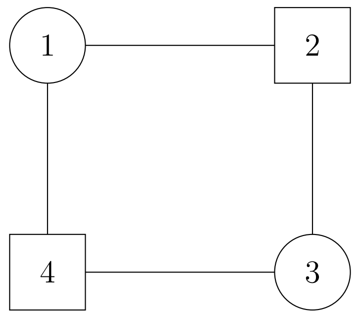
This general result applies to the case of a network of conformists, taking $\alpha_{ij} = 1$ if $\{i,j\} \in E$ and 0 otherwise, and $\theta_i = \mu_i|\Gamma_i|$, but it also applies to the case of a network where all agents are anti-conformist: just put $\alpha_{ij} = -1$ if $\{i,j\} \in E$ and 0 otherwise, and $\theta_i = -\mu_i|\Gamma_i| + 1$. Hence we have obtained:

Theorem 1. Suppose $N_c = \emptyset$ or $N_a = \emptyset$. Then the process converges to either an absorbing state or to a cycle of length 2.

The result is no more true if the network contains both conformists and anti-conformists, as the following example shows:

Example 1. Consider a graph with $n = 4$, where agents 1 and 3 are conformist, while 2 and 4 are anticonformist, situated as in the figure below, and take $\mu_i = 1/2$ for all $i \in N$. Then we have the following cycle of length 4:

$$(0,0,0,0) \to (0,1,0,1) \to (1,1,1,1) \to (1,0,1,0) \to (0,0,0,0).$$



2.3 Study of the complete network

We suppose in this section that the graph G is complete, i.e., every agent is connected to every other agent, so that the neighborhood Γ_i is N for every agent i.

We begin by recalling the classical result of Granovetter on absorbing states Granovetter (1978). Suppose $N = N_c$ and consider the cumulative distribution function F of the threshold of the agents:

$$F(x) = \frac{1}{n}|\{i \in N \,:\, \mu_i \leq x\}| = \frac{1}{n}\sum_{i\in N} \mathbb{1}_{\mu_i \leq x}.$$

This function is right-continuous, nondecreasing and has fixed points. It gives the proportion of agents whose threshold is below or equal to some quantity x, or put otherwise, the proportion of agents that will take action 1 when the current proportion of agents taking action 1 is x. As a consequence, if x^* is a fixed point of F, then $S^* := \{i \in N \,:\, \mu_i \leq x^*\}$ is an absorbing state, and conversely as well.

We generalize this result by incorporating anti-conformism as follows. We express by $G(x)$ the proportion of agents that will take action 1 when the current proportion of agents taking action 1 is x (hence $G(x) = F(x)$ is the cumulative distribution function of the threshold when there is no anti-conformists):

$$G(x) = \frac{1}{n}\Big(\sum_{i\in N_c} \mathbb{1}_{\mu_i \leq x} + \sum_{i\in N_a} \mathbb{1}_{\mu_i > x}\Big), \tag{3}$$

with $x \in [0, 1]$. This function, which we call the *transition function,* is still right-continuous but is no more nondecreasing in general, and as a consequence, does not necessarily have fixed points (see Figure 3). The following can be shown.

Theorem 2. The opinion dynamic converges to either absorbing states or cycles. It has absorbing state only when G has fixed point(s). The absorbing states of the dynamic process coincide with the fixed points of G as follows: if x^* is a fixed point of G, then

$$S^* = \{i \in N_c \,:\, \mu_i \leq x^*\} \cup \{i \in N_a \,:\, \mu_i > x^*\} \tag{4}$$

is an absorbing state, and vice versa each absorbing state S^* is associated with the fixed point $x^* = |S^*|/n$ of G. Which absorbing state can be reached is dependent on the initial state (when multiple absorbing states exist).

The theorem will be illustrated by several examples in the sequel. We being our study by supposing that the distribution of the threshold is uniform, then the Gaussian case and the general case will be studied.

2.3.1 Uniform distribution

The case of a uniform distribution permits to get explicit results. It has been studied by Granovetter Granovetter (1978), in order to explain riot phenomena (action 1: take part to a riot, action 0: be inactive). Supposing at the initial state that all agents are inactive, the presence of agents with threshold 0 (called "instigators" as they start rioting alone) initiates the phenomenon of rioting, which, by a domino or cascade effect, extends to the whole population if the distribution is uniform.

Specifically, we consider the thresholds are uniformly distributed over the set $\{0, 1/n, 2/n, \dots, n-1/n\}$, as in Granovetter (1978),[2] and that w.l.o.g. we may consider that agent 1 has threshold 0, agent 2 has threshold $1/n$, etc., and agent n has threshold $n-1/n$.[3] We denote by $\mu_\ell = \ell/n$ the threshold of agent $\ell + 1$.

Consider first that $N_a = \emptyset$. As expected, function G (which is in this case the cumulative distribution function F) has only one fixed point, which is $x^* = 1$, corresponding to the absorbing state $S^* = N$ (see Figure 1).

Introducing one anti-conformist agent Imagine now the conformist agent $k+1$ with threshold k/n becoming anti-conformist with the same threshold, denoted by $\mu_a = k/n$. Suppose for example that $n = 10$ and $k = 3$, which makes agent 4 to be anti-conformist (see Figure 2). According to Theorem 2, the absorbing states correspond to the fixed points of G, which are, in set notation:

$$\{1, 2, 3\},\ \{1, 2, 3, 5\},\ \{1, 2, 3, 5, 6\}, \dots, \{1, 2, 3, 5, 6, 7, 8, 9, 10\}.$$

Which absorbing state is reached depends on the initial condition. For example, starting from the state vector $(0, \dots, 0)$, agent 1 and 4 become active, which makes agent 2 and 3 to become active in addition, then agent 4 becomes inactive and no more changes occurs: the absorbing state $\{1, 2, 3\}$ has been reached. If now we start from the state vector $(1, \dots, 1)$, agent 4 becomes inactive but all the other remain active, so that the absorbing state $\{1, 2, 3, 5, 6, 7, 8, 9, 10\}$ is reached.

The following proposition summarizes the uniform case with one anti-conformist.

Proposition 1. Consider a group of agents whose thresholds follow a uniform distribution as described above, and suppose that there is only one anti-conformist, say agent $k + 1$, $k \in \{0, 1, \dots, n - 1\}$, with threshold $\mu_a := k/n$. Then the opinion dynamic has

[2] We may adopt another definition where the thresholds value from $1/n$ to 1 (note that there is no constant player then). As we will see below, there is no fundamental change in the results, except for the case $N_a = \emptyset$, where the domino effect would not start and $\emptyset$ would be the only absorbing state.

[3] We consider for ease of notation that only one agent has a given value of threshold. We may consider a more general situation where several agents have the same threshold. This will be considered in Section 2.3.3 with arbitrary distribution, however, in the case of a uniform distribution, this has no interest as uniformity obliges to have for each value of the threshold the same number of agents, so that everything goes exactly the same as the case of one agent per threshold value.

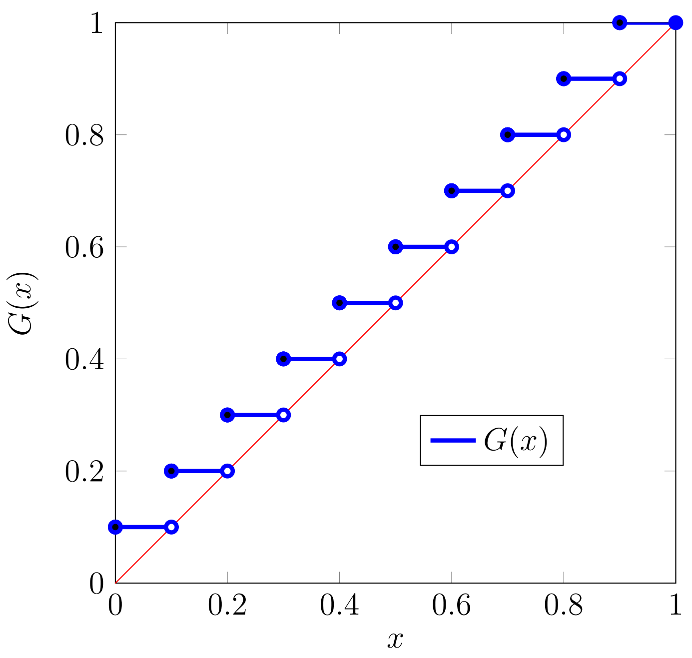


Figure 1: The transition function G for uniform threshold distribution function of conformists when $n = 10$

$n(1-\mu_a) = n-k$ absorbing states corresponding to the fixed points $k/n, \dots, n-1/n$ of G, specifically:

$$\{1,\dots,k\}, \{1,\dots,k,k+2\}, \{1,\dots,k,k+2,k+3\}, \dots, N \setminus \{k+1\}.$$

Moreover, starting from any initial state with the group opinion $x^* = k^*/n$, $k^* \in \{0,1,\dots,n\}$, if $k^* = n$, the reachable fixed point is $n-1/n$; if $k^* \geq k$ and $k^* \neq n$, the reachable fixed point is x^*; if $k^* < k$, then the reachable fixed point is k/n or $k+1/n$ depending on which has the same parity as k^*.

Introducing two anti-conformist agents Imagine now that two conformist agents, say k_1+1, k_2+1 with thresholds $\mu_a^1 = k_1/n, \mu_a^2 = k_2/n$ become anti-conformist with the same thresholds. Assum w.l.o.g. that $k_1 < k_2$.

Suppose for example that $n = 10$ and $k_1 = 3$ and $k_2 = 5$, which makes agent 4 and agent 6 to be anti-conformists (see Figure 3). According to Theorem 2, the dynamic has no absorbing states since there is no fixed points of G. Instead, there will be a cycle $S_1 \to S_2 \to S_1$ with $S_1 = \{1,2,3,5\}$, $S_2 = \{1,2,3,5,6\}$. For example, starting from the state vector $(0,\dots,0)$, agent 1, 4 and 6 become active, which makes agent 2 and 3 to become active in addition, then agent 4 becomes inactive, which will again activate agent 5 at the next stage i.e., the state $S_1 = \{1,2,3,5,6\}$ has reached with $x = 5/10$. Thus agent 6 becomes inactive at the next stage with the state $S_2 = \{1,2,3,5\}$ and the cycle $S_1 \to S_2 \to S_1$ has been reached.

The following proposition summarizes the uniform case with two anti-conformists.

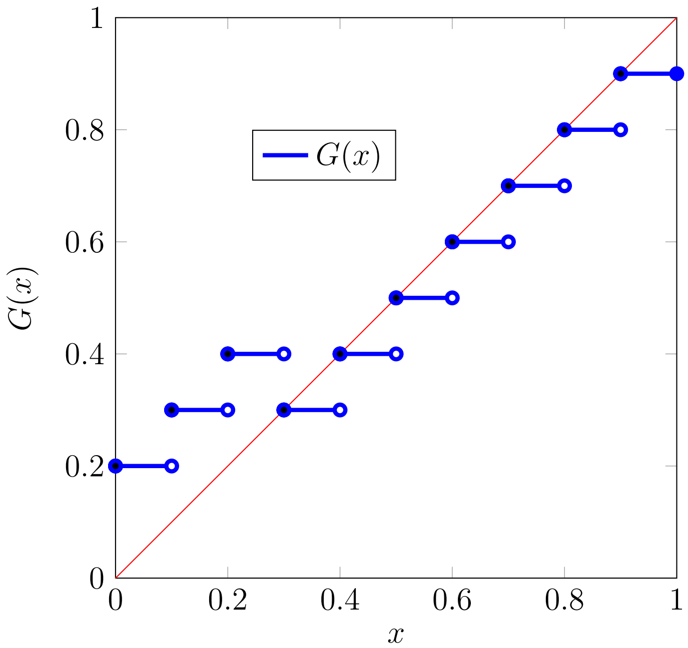


Figure 2: The transition function G for uniform threshold distribution function of conformists and anti-conformists when $k = 3$ and $n = 10$

Proposition 2. Consider a group of agents whose thresholds follow a uniform distribution as described above, and suppose that there are only two anti-conformists, say agents $k_1 + 1, k_2 + 1$ with thresholds $\mu_a^1 = {}^{k_1}/_n$ and $\mu_a^2 = {}^{k_2}/_n$, respectively[4], and without loss of generality, assume that $k_1 < k_2$. Then there is no absorbing state but a cycle $S_1 \to S_2 \to S_1$ with

$$S_1 = \{1, \dots, k_1, k_1 + 2, \dots, k_2\}, \quad S_2 = \{1, \dots, k_1, k_1 + 2, \dots, k_2 + 1\},$$

corresponding to group opinion (fixed points) ${}^{k_2}/_n \to {}^{(k_2-1)}/_n \to {}^{k_2}/_n$, regardless of the initial state.

The general case The previous results tend to indicate that with an odd number of types of anti-conformists, there are absorbing states, while there is no with an even number of types (only cycles occur). The main result of this section shows that this is indeed the case.

Theorem 3. Consider a society N of agents whose thresholds follow a uniform distribution on $\{0, {}^1/_n, \dots, {}^{n-1}/_n\}$ and suppose that some agents are anti-conformists ($N_a \neq \emptyset$). The following holds.

(i) Suppose that there are $2\ell + 1$ ($\ell \geq 1$) anti-conformist agents with thresholds $\mu_a^1, \cdots, \mu_a^{2\ell+1}$, respectively, with $\mu_a^i = {}^{k_i}/_n, (i = 1, \dots, 2\ell + 1)$ and $k_1 < k_2 < \cdots < k_{2\ell+1}$.[5] Then G has fixed points ${}^{k_{\ell+1}}/_n, \cdots, {}^{(k_{\ell+2}-1)}/_n$, whose corresponding

[4] $k_1, k_2 \in \{0, 1, \dots, n - 1\}$

[5] $k_1, k_2, \dots, k_{2l+1} \in \{0, 1, \dots, n - 1\}$

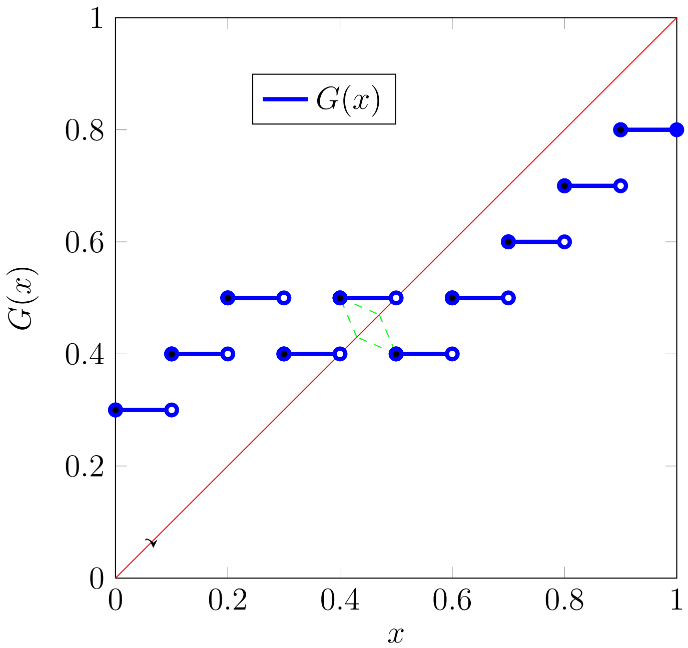


Figure 3: The transition function G for uniform threshold distribution function of conformists and anti-conformists when $k_1 = 3$, $k_2 = 5$ and $n = 10$. The cycle is materialized in green.

absorbing states are given by (4).

(ii) Suppose there are 2ℓ ($\ell \geq 1$) anti-conformist agents with thresholds $\mu_a^1, \cdots, \mu_a^{2\ell}$, respectively, with $\mu_a^i = {}^{k_i}/_{n}$, $(i = 1, \ldots, 2\ell)$ and $k_1 < k_2 < \cdots < k_{2\ell}$.[6] Then there is no absorbing state, but there exist cycles of length 2, corresponding to the pairs of points $(x, G(x)), (y, G(y))$ such that $G(y) = x$ and $y = G(x)$, i.e., x is a fixed point of $G^{(2)} = G \circ G$. Moreover, there is no cycle of length greater than 2.

When the uniform distribution is on $\{{}^1/_n, \ldots, 1\}$, it is easy to see that the results are the same as in Theorem 3, except that the cases of odd and even numbers of anti-conformists are inverted: there are absorbing states when there is an even number of conformists, and no absorbing states but cycles otherwise. This is because in that case, the function G is shifted of ${}^1/_n$ to the right, and $G(0) = p$, the number of anti-conformists.

Example 2. $n = 10$, $\ell = 2$, $k_1 = 0$, $k_2 = 1$, $k_3 = 2$, $k_4 = 3$ (see Figure 4). There is no absorbing state but cycles such as: $\{1, 2, 3\} \to \emptyset \to \{1, 2, 3\}$ with group opinion x: ${}^3/_{10} \to 0 \to {}^3/_{10}$. Other cycles might also exist, e.g., $\{3, 4\} \to \{4\} \to \{3, 4\}$ with group opinion x: ${}^1/_{10} \to {}^2/_{10} \to {}^1/_{10}$.

Example 3. $n = 10$, $\ell = 2$, $k_1 = 1$, $k_2 = 3$, $k_3 = 5$, $k_4 = 7$ (see Figure 5). There is no absorbing state but a cycle: $\{1, 3, 5, 6, 8\} \to \{1, 3, 5, 8\} \to \{1, 3, 5, 6, 8\}$ with group opinion s: ${}^5/_{10} \to {}^4/_{10} \to {}^5/_{10}$.

[6] $k_1, k_2, \ldots, k_{2l} \in \{0, 1, \ldots, n-1\}$

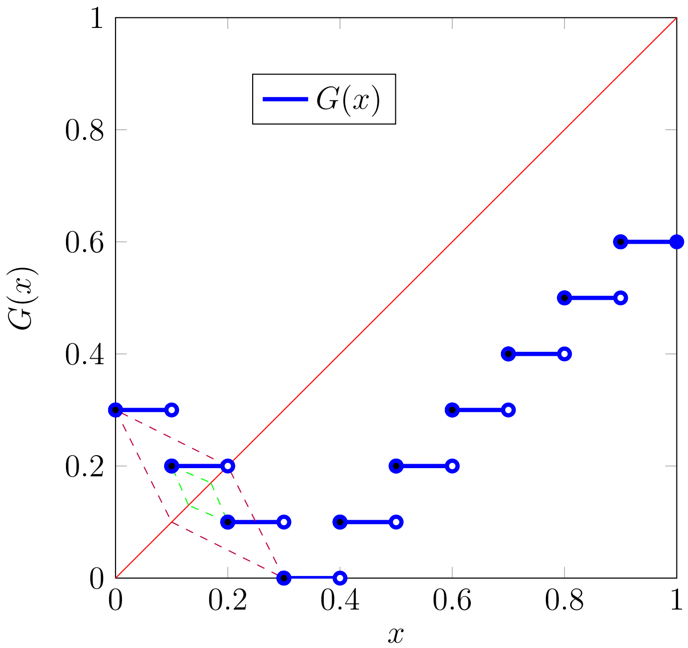


Figure 4: The transition function G for uniform threshold distribution function of conformists and anti-conformists when $k_1 = 0$, $k_2 = 1$, $k_3 = 2$, $k_4 = 3$ and $n = 10$. The cycles are materialized in red and green.

Applying the results above, we get the case where the society is purely anti-conformist.

Corollary 1. Consider a group of agents whose thresholds follow a uniform distribution. If all agents are anti-conformists, then there exist at most one absorbing state. The existence of absorbing state is decided by the parity of n. If n is even, then there is no absorbing state; if n is odd, then the absorbing state is the action profile associated to the fixed point $^{(n-1)}/_2$.

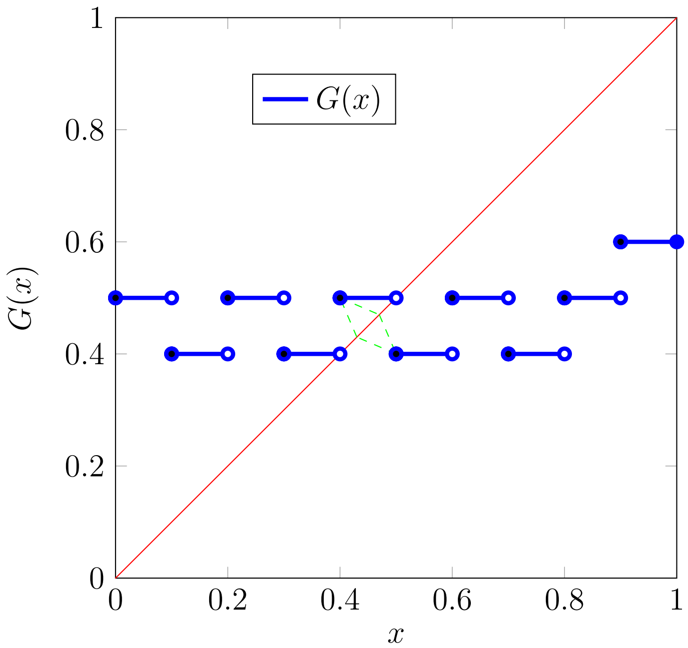


Figure 5: The transition function G for uniform threshold distribution function of conformists and anti-conformists when $k_1 = 1$, $k_2 = 3$, $k_3 = 5$, $k_4 = 7$ and $n = 10$. The cycle is materialized in green.

2.3.2 Gaussian distribution

In this section we explore the opinion dynamics for Gaussian distributed threshold model in complete networks by means of both simulations and analysis. Assume that the thresholds of conformists (respectively, anti-conformists) identically and independently distributed follow the Gaussian distributions $\mathbf{N}(m_c, \sigma_c)$, $\mathbf{N}(m_a, \sigma_a)$ respectively, with the corresponding cumulative distribution functions F_c, F_a. Then

$$G(x) = qF_c(x) + (1 - q)(1 - F_a(x))$$

where q is the proportion of conformists, and

$$F_c(x) = \frac{1}{\sigma_c\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{(t - m_c)^2}{2\sigma_c^2}\right) \mathrm{d}t = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x - m_c}{\sigma_c\sqrt{2}}\right)$$

and similarly for F_a, with $\mathrm{erf}(x)$ the error function defined by $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} \mathrm{d}u$. Letting $m_c = m_a =: m$ and $\sigma_c = \sigma_a =: \sigma$ permit to derive some properties. We obtain for the transition function

$$G(x) = \left(q - \frac{1}{2}\right) \mathrm{erf}\left(\frac{x - m}{\sigma\sqrt{2}}\right) + \frac{1}{2}, \tag{5}$$

from which we deduce that we always have $G(m) = 1/2$. Hence $1/2$ is a fixed point and therefore an absorbing state when $m = 1/2$. We also notice that when $\frac{x-m}{\sigma\sqrt{2}}$ tends to infinity (that is, when x tends to 1 and σ tends to 0), $G(x)$ tends to q (all conformists take action 1). Similarly, when $\frac{x-m}{\sigma\sqrt{2}}$ tends to $-\infty$ (i.e., x and σ tend to 0), $G(x)$ tends to $1-q$.

Let us examine if other absorbing states exist. First and second derivatives of the transition function are

$$G'(x) = \frac{2q-1}{2\sigma\sqrt{2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$$
$$G''(x) = -\frac{(2q-1)(x-m)}{2\sigma^3\sqrt{2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right).$$

The function has an inflection point at $x = m$. If $q < 1/2$, it is decreasing, concave when $x \leq m$, and convex when $x \geq m$. If $q = 1/2$, $G(x)$ is constant and equal to $1/2$, while if $q > 1/2$, it is increasing, convex when $x \leq m$, and concave when $x \geq m$.

Let us first consider the case $q < 1/2$ (majority of anti-conformists). G being decreasing and passing through the value $1/2$, it has a single fixed point x_* solution of the equation

$$\operatorname{erf}\left(\frac{x-m}{\sigma\sqrt{2}}\right) = \frac{1-2x}{1-2q}. \tag{6}$$

We have $m < x_* < 1/2$ when $m < 1/2$, and $1/2 < x_* < m$ when $m > 1/2$. Note that the convergence to x_* is done by doing large oscillations around x_*: again, the presence of anti-conformists slowers the convergence and make it more chaotic. However, no cycle can exist.

We consider now that $q > 1/2$ (majority of conformists). G being increasing makes the study more complex as several intersections with the diagonal may occur. As $\operatorname{erf}(x) \in]-1, 1[$, we have in general

$$1 - q < G(0) < G(m) = \frac{1}{2} < G(1) < q.$$

Therefore, if $m < 1/2$, there is a fixed point $x_* > 1/2$ solution of (6), and possibly another one smaller than $1/2$ solution of the same equation. If $m > 1/2$, the situation is symmetric: there is a fixed point $x_* < 1/2$ and possibly another one greater than $1/2$, both solutions of (6).

The case $m = 1/2$ is particular, because $1/2$ is a fixed point, and depending whether the tangent at $1/2$ is above or below the diagonal, there are two other fixed points, one greater than $1/2$ and the other one smaller, or no other fixed point. Observe that when $1/2$ is not the unique fixed point, it is unstable, as starting from state $x < 1/2$ (respectively, $> 1/2$) makes the process converge to the lower fixed point (respectively, the upper one). The condition for the tangent reads

$$G'(m) \geq 1 \Leftrightarrow \sigma \leq \frac{1}{\sqrt{2}}\left(q - \frac{1}{2}\right). \tag{7}$$

In any case, as G is increasing, the convergence to a fixed point is smooth and without oscillations. Note also that no cycle can occur.

We summarize our findings in the next proposition.

Proposition 3. Suppose that $m_a = m_c =: m$, $\sigma_a = \sigma_c =: \sigma$. Then no cycle can occur and an absorbing state always exists.

- If there are more conformists than anti-conformists ($q > 1/2$), there exists a fixed point $x_* \geq m$ if $m \leq 1/2$, or a fixed point $x_* \leq m$ if $m \geq 1/2$, both being solutions of (6). When $m = 1/2$, $1/2$ is a fixed point and two other fixed points exist, also solutions of (6), provided the variance is not greater than $\frac{1}{\sqrt{2}}\left(q - \frac{1}{2}\right)$. The convergence from a state x to x_* is done without oscillations, and the fixed point $\frac{1}{2}$, when it is not unique, is unstable.

- If there are more anti-conformists than conformists ($q < 1/2$), there is a unique fixed point x_* given by solving (6). The convergence from a state x to x_* is done with oscillations around x_*.

- If there are exactly as many conformists as anti-conformists, then there is convergence in one shot from any state x to $1/2$.

The following examples illustrate the above results.

Example 4. $q = 0.9$; $m = 0.5$; $\sigma = 0, 1, 0.2, \ldots, 0.9$ (see Figure 6). This is the case of a majority of conformists and $m = 1/2$. One can observe the fixed point $1/2$ and the possible existence of 2 others. The limit value of σ for the tangent condition (7) is 0.283. Also, one can observe the asymptotic values q and $1 - q$ for $G(x)$.

Example 5. $q = 0.95$; $m = 0.3$; $\sigma = 0, 1, 0.2, \ldots, 0.9$ (see Figure 7). This is the case of a majority of conformists and $m < 1/2$. There is fixed point greater than $1/2$, whose value is negatively related to the variance, and two other possible ones smaller than $1/2$.

Example 6. $q = 0.9$; $m = 0.8$; $\sigma = 0, 1, 0.2, \ldots, 0.9$ (see Figure 8): majority of conformists and $m > 1/2$. There is fixed point smaller than $1/2$, whose value is positively related to the value of the variance.

Example 7. $q = 0.1$; $m = 0.2$; $\sigma = 0, 1, 0.2, \ldots, 0.9$ (see Figure 9). This is the case of a majority of anti-conformists and $m < 1/2$. There is fixed point smaller than $1/2$. Its value is positively related to the value of the variance.

2.3.3 General distribution

Recall that if $N_a \neq \emptyset$, the function G may not have fixed points because the presence of anticonformists makes it nonmonotonic. It is therefore difficult to get precise results in the general case. The next proposition elucidates the situation when there is only one type of anti-conformist agent.

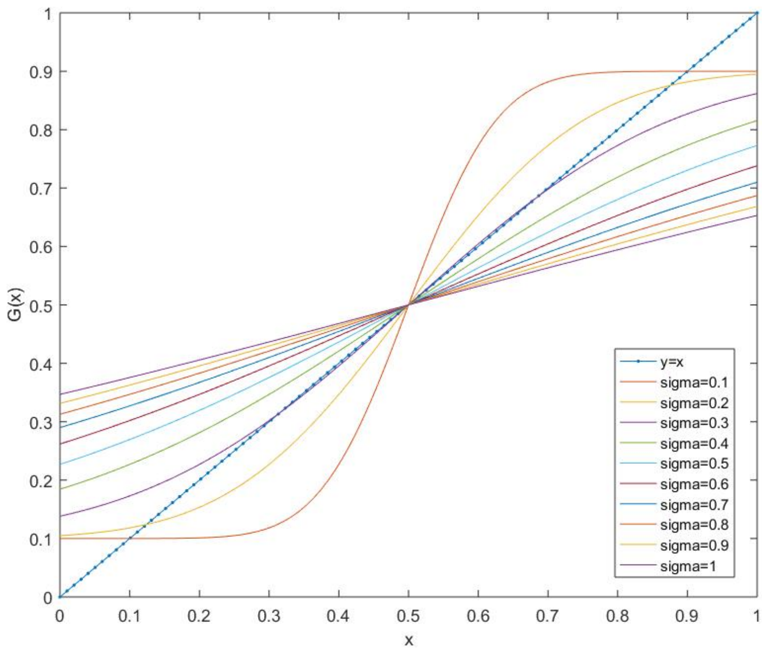


Figure 6: Gaussian distribution with a small proportion of anti-conformist agents ($q = 0.9$), $m = 0.5$

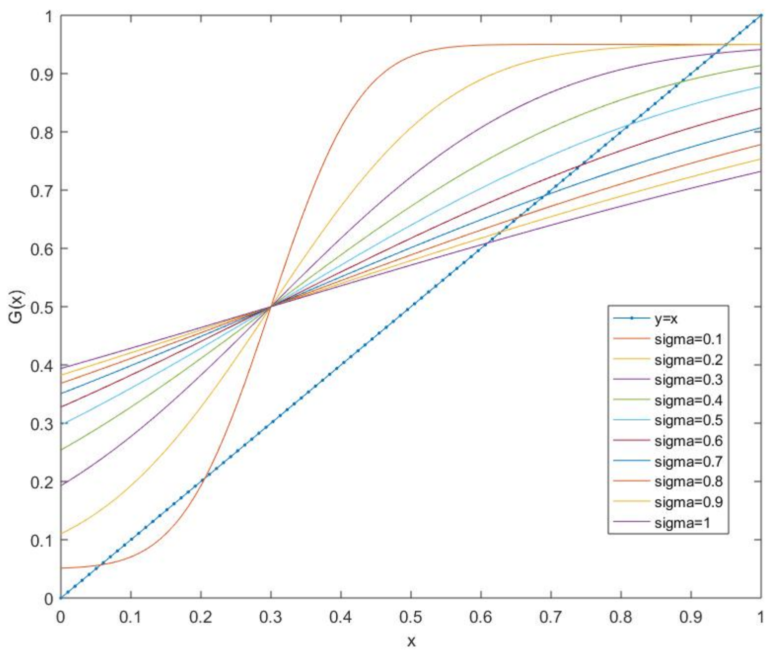


Figure 7: Gaussian distribution with a small proportion of anti-conformist agents ($q = 0.95$), $m = 0.3$

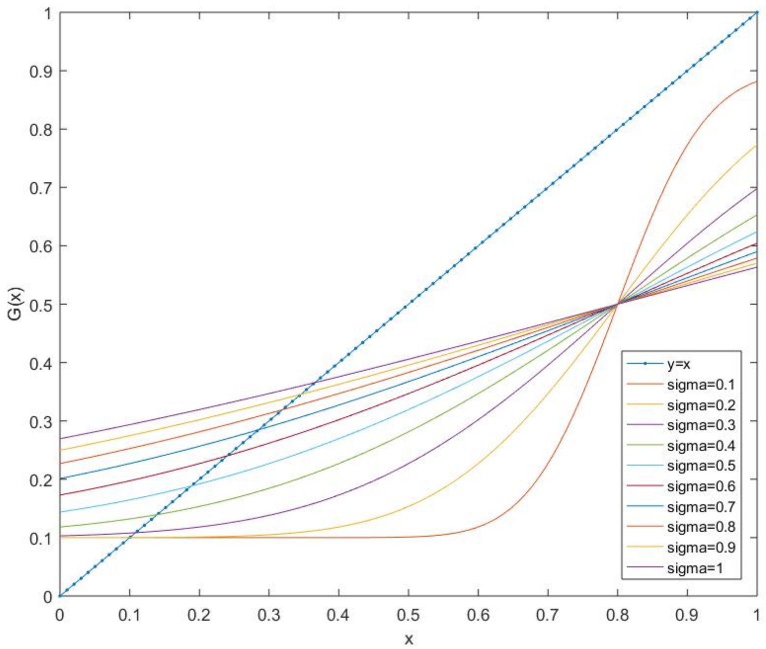


Figure 8: Gaussian distribution with a small proportion of anti-conformist agents ($q = 0.9$), $m = 0.8$

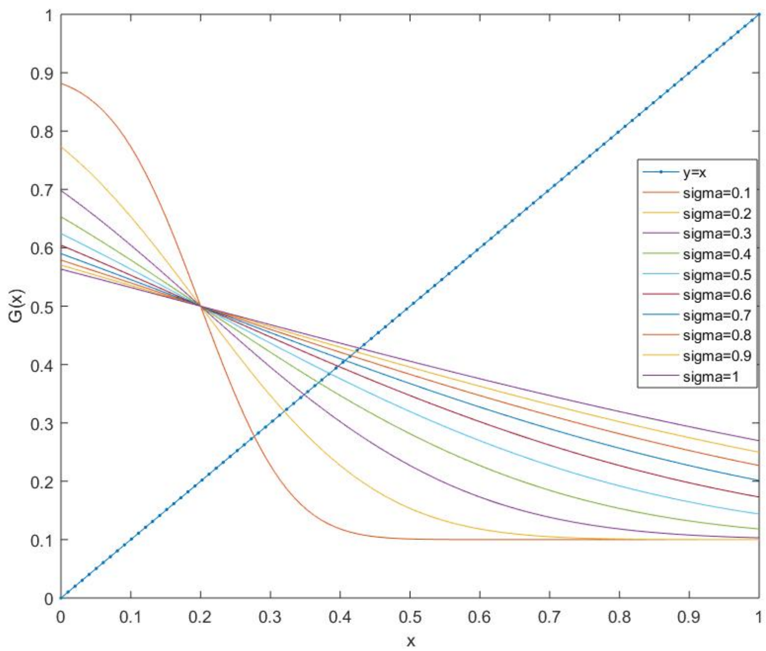


Figure 9: Gaussian distribution with a large proportion of anti-conformist agents ($q = 0.1$), $m = 0.2$

Proposition 4. Consider the following case where there is only one type of anti-conformists with threshold μ_a and assume its fraction among all players is δ_a. The list of all threshold values of conformists are $\mu_1, \mu_2, \cdots, \mu_p$ (in a strict increasing order) with fractions $q_1, q_2, \cdots, q_p$ respectively. Denote by k the largest number such that $\mu_k \leq \mu_a$ (see Table 1). The following holds.

(i) If there is no agent i such that $\mu_i = \mu_a$, then there is no absorbing state if and only if the thresholds and corresponding fractions satisfy the following inequalities:[7]

$$\begin{cases} \delta_a \geq \mu_1 \\ \delta_a + \sum_{i=1}^{i_0} q_i \geq \mu_{i_0+1} \qquad (i_0 = 1, 2, \ldots, k-1) \\ \delta_a + \sum_{i=1}^{k} q_i \geq \mu_a \\ \sum_{i=1}^{k} q_i < \mu_a \\ \sum_{i=1}^{i_0} q_i < \mu_{i_0} \qquad (i_o = k+1, k+2, \ldots, p) \end{cases} \tag{8}$$

(ii) Otherwise, there is no absorbing state if and only if the thresholds and corresponding fractions satisfy the following inequalities:[8]

$$\begin{cases} \delta_a \geq \mu_1 \\ \delta_a + \sum_{i=1}^{i_0} q_i \geq \mu_{i_0+1} \quad (i_0 = 1, 2, \ldots, k-1) \\ \sum_{i=1}^{i_0} q_i < \mu_{i_0} \qquad (i_o = k, k+1, \ldots, p) \end{cases} \tag{9}$$

Proposition 4 gives the necessary and sufficient conditions by a system of inequalities such that no absorbing state exists for general distributions when there is only one type

[7]Note that if $\mu_a < \mu_1$, we can think it as $k = 0$ and delete all the terms related to non positive indices. Thus inequalities (8) become

$$\begin{cases} \delta_a \geq \mu_a \\ \sum_{i=1}^{i_0} q_i < \mu_{i_0} \quad (i_0 = 1, 2, \ldots, p) \end{cases}$$

[8]Note that if $\mu_a = \mu_1$, we can think it as $k = 1$ and delete all the terms related to non positive indices. Thus inequalities (9) will be

$$\begin{cases} \delta_a \geq \mu_1 \\ \sum_{i=1}^{i_0} q_i < \mu_{i_0} \quad (i_0 = 1, 2, \ldots, p) \end{cases}$$

Table 1: Distribution of agents' thresholds with conformists and one type of anti-conformists

Proportion	Threshold	Behavior characteristics
q_1	μ_1	Conformism
$\cdots$	$\cdots$	$\cdots$
q_k	μ_k	Conformism
δ_a	μ_a	Anti-conformism
q_{k+1}	μ_{k+1}	Conformism
$\cdots$	$\cdots$	$\cdots$
q_p	μ_p	Conformism

of anti-conformist agent. Figure 10 gives the possible absorbing states for different ranges of μ_a and δ_a based on Proposition 4 and its proofs. In the left area filled with only north west lines, i.e. $\mu_a \leq \sum_{i=1}^{k} q_i$, the opinion dynamic has only one absorbing state where only conformists with thresholds $\mu_1, \dots, \mu_k$ saying yes, i.e., $x = \sum_{i=1}^{k} q_i$. In the right area filled with only north east lines, i.e., $\delta_a < \mu_a - \sum_{i=1}^{k} q_i$, the opinion dynamic has only one absorbing state where the anti-conformists and conformists with thresholds $\mu_1, \dots, \mu_k$ saying yes, i.e., $x = \delta_a + \sum_{i=1}^{k} q_i$. In the bottom area filled with only horizontal lines, there are three possible absorbing states. The overlapped area means that absorbing states of both cases are possible. All the details can be found in Proof of Proposition 4.

The following example illustrates the case where there is no absorbing state.

Example 8. We consider $n = 10$, with $N_c = \{1, 2, 3, 4, 5, 6\}$ and 4 anti-conformists. The parameters are $\mu_a = \delta_a = 4/10$, $\mu_1 = q_1 = q_2 = q_3 = q_4 = q_5 = q_6 = 1/10$, $\mu_2 = 2/10$, $\mu_3 = 3/10$, $\mu_4 = 5/10$, $\mu_5 = 6/10$, $\mu_6 = 7/10$ (see Figure 11). There is no absorbing state but a cycle: $\{1, 2, 3, 4, 5, 6\} \to \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4\} \to \{1, 2, 3\} \to \{1, 2, 3\} \cup N_a \to \{1, 2, 3, 4, 5, 6\}$ with group opinion x: $6/10 \to 5/10 \to 4/10 \to 3/10 \to 7/10 \to 6/10$.

The previous example has shown the existence of cycles. The next theorem establishes that there could be at most one cycle, whose length has an upper bound.

Theorem 4. Consider the same assumptions and notation as in Proposition 4. Then the opinion dynamic has either absorbing states or a unique cycle of length at most $m + 2$, where m is the number of values μ_i in the interval $\left]\sum_{i=1}^{k} q_i, \sum_{i=1}^{k} q_i + \delta_a\right]$. The upper bound of the length of the cycle, considering any possible values for the thresholds and fractions, is $n_a + 1$, where $n_a = n\delta_a$ is the number of anti-conformists.

The next example illustrates the theorem and shows that the cycle can be shorter than $m + 2$ and that its length can be far below the upper bound $n_a + 1$. Note that in Example 8, this bound is attained.

Example 9. We consider $n = 100$, with the following parameters: $\delta_a = 0.4$ (40 anti-conformists), $\mu_a = 0.5$, $\mu_1 = 0,2$, $q_1 = 0.1$, $\mu_2 = 0.3$, $q_2 = 0.15$, $\mu_3 = 0.7$, $q_3 = 0.3$, $\mu_4 = 0.8$, $q_4 = 0.05$. Then $k = 2$, and by the theorem, the cycle should be of length at most 3, while the upper bound $n_a + 1$ yields 41. On sees on Figure 12 that the cycle (in green) has in fact length 2 and is formed of the two points $(0.25, 0.5), (0.5, 0.25)$.

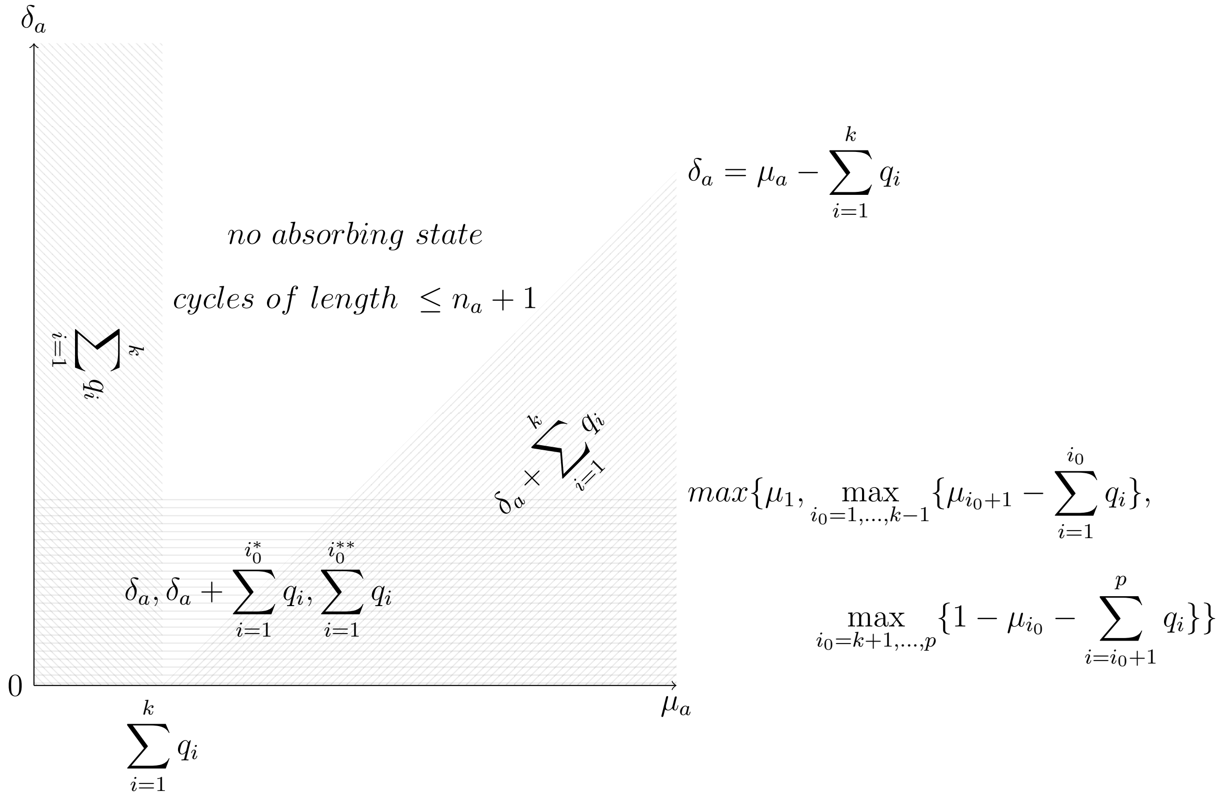


Figure 10: Existence of absorbing states for general distributions when there is only one type of anti-conformist agent with threshold μ_a and proportion δ_a (the variables in each area filled with lines indicates the values of the possible fixed points corresponding to each case.)

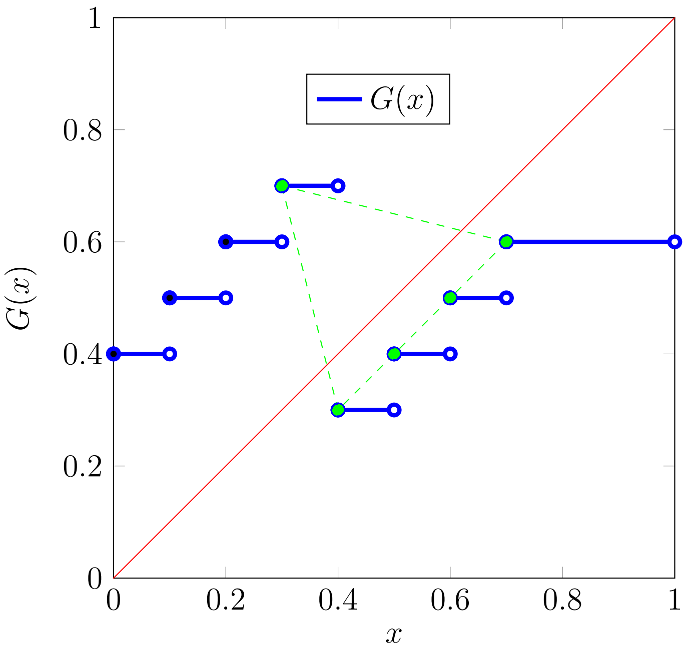


Figure 11: $G(x)$ of Example 8, with a cycle of length 5

3 Random sampling models

The previous section considered a mechanism of diffusion with a complete and undirected network, where each agent was permanently in contact with all other agents. As this assumption may be unrealistic in some situations, we consider here a different mechanism where agents meet other agents at random (random neighborgood), with a certain size of the neighborhood to be either fixed or drawn from a distribution. When the size is fixed and identical for all agents, we speak of a *homogeneous network*. Note that in this model we implicitly assume that the network is directed, as if i selects j in its neighborhood, this does not necessarily imply that j has i in its neighborhood. Also, note that this model amounts to considering that at each time step a random graph on N realizes, where each agent has a random neighborhood (and therefore a random degree). We denote by $P(k)$ the distribution of the degree of each agent, who are supposed to have the same distribution.

To avoid intricacies, it is convenient to consider that the random neighborhood of agent i, knowing that its degree is d, is taken as a random subset of N, of size d. This means that sometimes i is in its neighborhood, sometimes it is not. Still, the agents are considered to have a threshold, which can be drawn from a distribution or is fixed.

An important consequence of the model is that the process of updating of the opinion is no more deterministic, but still obeys a Markov chain. Its analysis is therefore much more complex, as not only absorbing states and cycles can exist but also aperiodic and periodic absorbing classes, where a class is a set of states such that a chain of transitions exists from any state to any other state in the set, and which is maximal for this property. It is easy to see that a state T different from $\emptyset$ and N cannot be absorbing anymore: this is because the neighborhood being random and smaller than N, it is not guaranteed

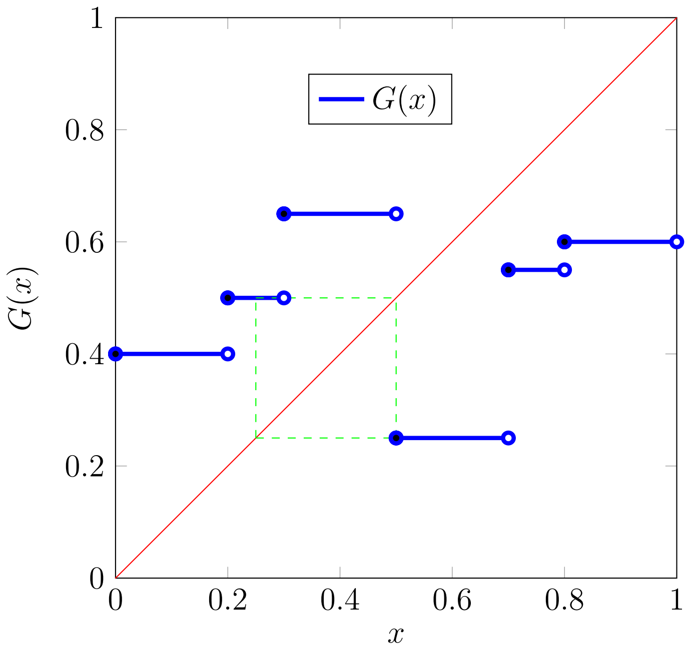


Figure 12: $G(x)$ of Example 9, with a cycle of length 2

that it will contain T at each period. However, $\emptyset$ and N can still be absorbing. The next lemma clarifies this point.

Lemma 1. $\emptyset$ is absorbing (resp., N is absorbing) iff all anti-conformists are constant 0-players (resp., constant 1-players), while there is no constant conformist player (i.e., $0 < \mu_i \leq 1$ for all $i \in N_c$).

Proof. Suppose $T = \emptyset$. As $s_i(\emptyset) = 0$ with certainty for every i, by the assumption every conformist will take action 0 with certainty. Now, $i \in N_a$ takes action 0 iff $\mathbb{P}(0 \geq \mu_i) = 1$, i.e., $\mu_i = 0$. Hence, any anti-conformists must be a constant 0-player.

The argument for $T = N$ is much the same. □

The existence of non trivial absorbing classes will be shown in Section 3.1.1, where a complete analysis is done in a simple case (only two different thresholds, one for conformists and one for anti-conformists). The complexity of the results shows that it seems out of reach to get a complete study in more general cases. Nevertheless, general results, although not exhaustive, can be obtained (see Section 3.1.2).

We start by focusing on the case of fixed degree (homogeneous networks).

3.1 Homogeneous networks

We suppose in this section that the neighborhood of every agent has a fixed size d. A complete study of this case is possible when all conformist agents have the same threshold μ_c, and all the anti-conformist agents have threshold μ_a. Then we give a result in the general case. We begin by some general considerations.

Let us express the probability of transition $\mathbb{P}(S \to T)$ from one state S to another state T. We have by the independence assumption that $\mathbb{P}(S \to T) = \prod_{i \in T} p_i^1(S) \prod_{i \notin T} p_i^0(S)$, where $p_i^e(S)$ is the probability for agent i to take action $e \in \{0,1\}$ knowing that the current state is S. To compute these probabilities, it is necessary to compute the distribution of the average opinion $\overline{a_i}$ in the neighborhood of i knowing the current state S. It is easy to check that

$$\mathbb{P}(\overline{a_i} = \frac{k}{d} \mid S) = \begin{cases} \frac{\binom{s}{k}\binom{n-s}{d-x}}{\binom{n}{d}}, & \text{if } d-n+s \leq k \leq s \\ 0, \text{otherwise}, & \end{cases} \tag{10}$$

for $k = 0, 1, \ldots, d$. Observe that these probabilities do not depend on i, therefore we can omit the subscript i and write $\overline{a}$, the average opinion in a neighborhood. Then,

$$\text{If } i \in N_c, \ \ p_i^1(S) = \mathbb{P}(\overline{a} \geq \mu_i \mid S) \text{ and } p_i^0(S) = 1 - p_i^1(S) \tag{11}$$

$$\text{If } i \in N_a, \ \ p_i^1(S) = \mathbb{P}(\overline{a} < \mu_i \mid S) \text{ and } p_i^0(S) = 1 - p_i^1(S). \tag{12}$$

As these probabilities depends only on the cardinality of S, we may write $p_i^1(s)$ for simplicity.

3.1.1 Case with two thresholds μ_a, μ_c

We assume here that there are two types of agents: anti-conformist agents with threshold μ_a and conformist agents with threshold μ_c, where $0 < \mu_a, \mu_c \leq 1$.

Observe that $p_i^1(s)$ depends only on whether i belongs to N_a or N_c. Specifically, for a conformist agent i, $p_i^1(S) = \mathbb{P}(\overline{a} \geq \mu_c \mid S)$ is a nondecreasing function of $s = |S| \in \{0, 1, \ldots, n\}$ to $[0,1]$, depending only on μ_c, n and d. In addition, we have $p_i^1(0) = 0$ and $p_i^1(n) = 1$. Similarly, if i is anti-conformist, $p_i^1(s)$ is a nonincreasing function of s, starting at 1 with $s = 0$ and finishing at 0 with $s = n$. Thus, we fall into the framework studied in Grabisch et al. (2018) on an anonymous model of anti-conformism where each agent i has the probability $p_i(s)$ to take action 1 at next step knowing that the current state is S, with $s = |S|$, and $p_i(s)$ is a nondecreasing (respectively, nonincreasing) function reaching values 0 and 1 when i is conformist (respectively, anti-conformist).

In the model of Grabisch et al. (2018), all functions p_i can be different among the agents, but it is required that for all conformists, the functions p_i have the same domain where they take value 0 and 1, and similarly for the anti-conformists. These domains are characterized for the conformists by the quantities l^c (firing threshold) and $n - r^c$ (saturation threshold) which are the rightmost and leftmost values of s for which $p_i^1(s)$ is 0 and 1, respectively, given by

$$l^c := \min\{s : p_i(s) > 0\} - 1, \quad n - r^c = \min\{s : p_i(s) = 1\},$$

and similarly for the anti-conformists:

$$l^a := \min\{s : p_i(s) < 1\} - 1, \quad n - r^a = \min\{s : p_i(s) = 0\}.$$

Our case satisfies these requirements as $p_i^1(s)$ depends only on μ_a, μ_c and d. We easily obtain:

$$\begin{aligned}
l^c &= \max_{i \in N_c} \{s \mid \overline{a} < \mu_c\} = \lceil d\mu_c \rceil - 1 \\
n - r^c &= \min_{i \in N_c} \{s \mid \overline{a} \geq \mu_c\} = n - d + \lceil d\mu_c \rceil \\
l^a &= \max_{i \in N_a} \{s \mid \overline{a} < \mu_a\} = \lceil d\mu_a \rceil - 1 \\
n - r^a &= \max_{i \in N_a} \{s \mid \overline{a} \geq \mu_a\} = n - d + \lceil d\mu_a \rceil .
\end{aligned}$$

From Grabisch et al. (2018), we know that in full generality 19 possible absorbing classes can occur, depending on the values of l^c, r^c, l^a, r^a. Since it holds that in our case $l^c + r^c = l^a + r^a = d - 1 < n - 1$, 4 among the 19 are not possible. Denoting by n_c the number of conformist agents, we give below the list of the remaining 15 absorbing classes, put into categories.

Polarization: the society of agents is divided in two groups, one taking action 1, the other taking action 0.

(1) N_a if and only if $n_c \geq \max\{n - l^c, n - l^a\}$;

(2) N_c if and only if $n_c \geq \max\{n - r^c, n - r^a\}$;

Cycles: sequence of states made of the infinite repetition of a pattern.

(3) $N_a \xrightarrow{1} \emptyset \xrightarrow{1} N_a$ if and only if $n - l^c \leq n_c \leq r^a$;

(4) $N_c \xrightarrow{1} N \xrightarrow{1} N_c$ if and only if $n - r^c \leq n_c \leq l^a$;

(5) $N_a \xrightarrow{1} N_c \xrightarrow{1} N_a$ if and only if $n_c \leq \min\{l^c, l^a, r^c, r^a\}$;

(6) $\emptyset \xrightarrow{1} N_a \xrightarrow{1} N_c \xrightarrow{1} \emptyset$ if and only if $n_c \leq \min\{r^c, r^a, l^c\}$ and $n_c \geq n - r^a$;

(7) $N_a \xrightarrow{1} N \xrightarrow{1} N_c \xrightarrow{1} N_a$ if and only if $n_c \leq \min\{l^c, l^a, r^c\}$ and $n_c \geq n - l^a$;

Fuzzy cycles: the pattern contains states but also intervals of states. This means that there is no exact repetition of the same pattern, but at each repetition a state is picked at random in the interval.

(8) $N_a \xrightarrow{1} [\emptyset, N_c] \xrightarrow{1} N_a$ if and only if $n_c \leq \min\{l^c, l^a, r^a\}$ and $r^c < n_c < n - l^c$;

(9) $N_c \xrightarrow{1} [N_a, N] \xrightarrow{1} N_c$ if and only if $n_c \leq \min\{r^c, r^a, l^a\}$ and $l^c < n_c < n - r^c$;

(10) $[\emptyset, N_c] \xrightarrow{1} [N_a, N] \xrightarrow{1} [\emptyset, N_c]$ if and only if $\max\{r^c, l^c\} < n_c \leq \min\{r^a, l^a, n - l^c - 1, n - r^c - 1\}$;

Fuzzy polarization: the polarization is defined by an interval, which means that at each time step, a state is picked at random in the interval, representing the set of active agents.

(11) $[\emptyset, N_a]$ if and only if $\max\{n - l^c, r^a + 1\} \leq n_c < n - l^a$;

(12) $[N_c, N]$ if and only if $\max\{n - r^c, l^a + 1\} \leq n_c < n - r^a$;

Chaotic polarization: similar to the previous case but more complex as several intervals are involved.

(13) $[\emptyset, N_a] \cup [\emptyset, N_c]$ if and only if $l^c \geq n - r^a$ and $n_c \in (]r^c, n - l^c[\cap]l^a, n - r^c[) \cup ((]l^a, n - r^a[\cup]l^c, n - r^c[)\cap]0, r^c[)$;

(14) $[N_a, N] \cup [N_c, N]$ if and only if $l^a \geq n - r^c$ and $n_c \in (]l^c, n - r^c[\cap]r^a, n - l^c[) \cup ((]r^a, n - l^a[\cup]r^c, n - l^c[)\cap]0, l^c[)$;

Chaos: at each time step a state is picked at random among all possible states.

(15) 2^N otherwise.

When n tends to infinity Assume that the number of agents tends to infinity. For simplicity, divide the previous parameters $n^a, n^c, l^a, l^c, r^a, r^c, d$ by n, keeping the same notation so that these parameters now take value in $[0,1]$. Thus $l^c = d\mu_c$ and $r^c = d(1-\mu_c)$, and smilarly for l^a, r^a.

We examine different typical situations for the value of the parameters, taking advantage of the study made in Grabisch et al. (2018).

- **Situation 1:** $l^a = l^c =: l$ and $r^a = r^c =: r$. This implies $\mu_c = \mu_a =: \mu$. Only the following four absorbing classes remain possible in this situation:

  - N^a iff $n^a \leq l$
  - N^c iff $n^a \leq d - l = d(1-\mu)$
  - cycle $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$ iff $n^a \geq 1 - d\mu$ and $n^a \geq 1 - d(1-\mu)$
  - 2^N otherwise

  A cascade effect occurs in this situation when the proportion of anti-conformist agents is not too large. When the firing threshold is very low, it will lead to a cascade with all conformist agents saying yes; when the firing threshold is very high, it will lead to a cascade with all anti-conformist agents saying yes. As the proportion of anti-conformist agents increases, the society goes from consensus, to polarization or cascade, then to a chaos, finally to a cycle.

- **Situation 2:** $l^a = l^c = r^a = r^c = \frac{d}{2}$. This implies $\mu_c = \mu_a = 1/2$. The three possible absorbing classes in this situation are:

  - N^a, N^c iff $n^a \leq d/2$ (”polarization”)
  - cycle $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$ iff $n^a \geq 1 - d/2$ (”cycle”)
  - 2^N otherwise (”chaos”)

  The possible absorbing classes of ”fuzzy cycle” and ”fuzzy polarization” mentioned in Grabisch et al. (2018) become impossible since there is a constraint $l^a + r^a = l^c + r^c$ in this special context. Note that polarizations N^c and N^a always appear together, implying that there is no cascade effect.

- **Situation 3:** n^a tends to 0. Assume that $n^a = \epsilon > 0$ arbitrarily small, therefore $n^c = 1 - \epsilon$.

  Among the initial 15 possible absorbing classes, only 7 of them remain possible:

  - (1) N^a iff l^a or $l^c \geq \epsilon$;
  - (2) N^c iff r^c or $r^a \geq \epsilon$’
  - (3) $N^a \xrightarrow{1} \emptyset \xrightarrow{1} N^a$ iff $l^c \geq \epsilon$ and $r^a \geq 1 - \epsilon$;
  - (4) $N^c \xrightarrow{1} N \xrightarrow{1} N^c$ iff $r^c \geq \epsilon$ and $l^a \geq 1 - \epsilon$;
  - Classes (5) to (10) are impossible;
  - (11) $[\emptyset, N^a]$ iff $l^a < \epsilon$, $l^c \geq \epsilon$ and $r^a < 1 - \epsilon$;

– (12) $[N^c, N]$ iff $r^a < \epsilon$, $r^c \geq \epsilon$ and $l^a < 1 - \epsilon$;
– Classes (13) and (14) are impossible;
– (15) 2^N otherwise.

Again there is no cascade effect in this situation since two possible polarizations always appear together.

3.1.2 General case

When the distribution of thresholds is arbitrary, we show that in most cases, only chaos can occur, i.e., the only absorbing class is 2^N.

Theorem 5. Suppose $n_a \geq d$, $n_c \geq d$ and suppose that there is no constant player (i.e., $0 < \mu_i \leq 1$ for every player i). Then 2^N is the only absorbing class, i.e., the transition matrix is irreducible.

Corollary 2. Suppose the distribution of thresholds has support $\{1/d, \ldots, 1\}$ for the conformists and the anti-conformists. Then 2^N is the only absorbing class.

Indeed, the assumption implies that there are at least d members in N_a, N_c.

3.2 Arbitrary degree distribution

We suppose now that the degree of the neighborhood is not fixed but follows a distribution $P(d)$. We try to generalize the results of the homogeneous case.

The probabilities of taking action 1 or 0 for the conformist and anti-conformist agents given in (11) and (12) become:

$$\text{If } i \in N_c,\ \ p_i^1(S) = \sum_d \mathbb{P}(\overline{a}_i \geq \mu_i \mid S; d)P(d) \text{ and } p_i^0(S) = 1 - p_i^1(S) \tag{13}$$

$$\text{If } i \in N_a,\ \ p_i^1(S) = \sum_d \mathbb{P}(\overline{a}_i < \mu_i \mid S; d)P(d) \text{ and } p_i^0(S) = 1 - p_i^1(S), \tag{14}$$

where the summation over d is taken over the support of $P(d)$, and $\mathbb{P}(\overline{a}_i \geq \mu_i \mid S; d)$ is given by (10).

3.2.1 Case with two thresholds μ_a, μ_c

The introduction of a distribution over the degree does not change the behavior of $p_i^1(S)$: there are still nonincreasing or nondecreasing functions of s taking boundary values 0 and 1. The identification of the absorbing classes depends only on the width of the domain where these functions take values 0 and 1, hence their exact form is unimportant for this purpose.

By (13) we see that $p_i^1(S) = 0$ for conformist agents iff every term in the summation is equal to 0. We have established in Section 3.1.1 that $\mathbb{P}(\overline{a}_i \geq \mu_i \mid S; d) = 0$ iff $s \leq l^c(d_i) := \lceil d_i \mu_c \rceil - 1$ and $\mathbb{P}(\overline{a}_i \geq \mu_i \mid S; d) = 1$ iff $s \geq n - r^c(d_i) := n - d_i + \lceil d_i \mu_c \rceil$. Introducing

$$l^c := \min\{l^c(d) \,:\, d \in \text{ support of } P(d)\}, r^c = \min\{r^c(d) \,:\, d \in \text{ support of } P(d)\}$$
$$l^a := \min\{l^a(d) \,:\, d \in \text{ support of } P(d)\}, r^a = \min\{r^a(d) \,:\, d \in \text{ support of } P(d)\},$$

the results of Section 3.1.1 can be readily extended to the general case by using the above quantities l^c, r^c in all the conditions of existence of the 15 absorbing classes.

An important consequence is the following: suppose that the distribution of d gives a positive probability to $d = 1$. Then we find $l^c = r^c = l^a = r^a = 0$. By inspection of the conditions of existence of the 15 absorbing classes, it follows that *only* the case of the chaos (2^N) remains possible. Note that this assumption is often satisfied (e.g., for the Poisson distribution, which arises when any pair of vertices is connected with a fixed probability).

3.2.2 General case

We suppose now that each agent has a fixed threshold but possibly different among agents. A generalization of Theorem 5 is possible: under mild assumptions, only chaos can occur. Let us denote by $\underline{d}, \overline{d}$ the lowest and greatest values of d with a positive probability, and by $\underline{\mu}, \overline{\mu}$ the lowest and highest threshold values among the agents.

Theorem 6. Suppose that $\underline{\mu} > 0$ (no constant player) and that the number of conformist and anti-conformist agents satisfy

$$\overline{\mu}\underline{d} \leq n_a, \ n_c < n - \underline{d}(1 - \underline{\mu})$$

Then 2^N is the only absorbing class, i.e., the transition matrix is irreducible.

Note that the conditions on n_a, n_c can be written equivalently as $n_a \geq \overline{\mu}\underline{d}$ and $n_a > \underline{d}(1 - \underline{\mu})$ (same for n_c). Again, observe that if $\underline{d} = 1$, these conditions are *always* satisfied.

4 Concluding remarks

In this paper we explore the threshold model of collective behavior with both conformist and anti-conformist agents.

Firstly, we study the threshold model in a fixed complete network, supposing a uniform distribution of the threshold, a Gaussian distribution and arbitrary distributions with one type of anti-conformist, respectively. Within the structure of a complete network, the opinion dynamic converges to either absorbing states or cycles. Whether there are absorbing states or cycles depends on the existence of fixed points of the transition function. For a uniform distributed threshold model, we show that if there are odd number of anti-conformist agents, then the opinion dynamic has absorbing states coinciding with the fixed points of the transition function; if there are even number of anti-conformist agents, then there is no absorbing states but cycles of length 2. For Gaussian distributed threshold model, the opinion dynamic has no cycle but absorbing states which are the solutions of equation (6). For more general distributed threshold model, we have given the necessary and sufficient conditions such that no absorbing state exists but a cycle. Furthermore, we also provide an upper bound for the length of the cycle.

Secondly, we study the threshold model in a random neighborhood network where the neighborhood of an agent is random, drawn at each tie step from a distribution. Supposing a homogeneous network where all agents have a fixed degree, 15 possible absorbing classes can occur in the case with two thresholds μ_a and μ_c, including polarization, cycles, fuzzy cycles, fuzzy polarization, chaotic polarization and even chaos. For arbitrary degree distributions, under mild assumptions, only chaos can occur.

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A Proof of Theorem 2

The opinion dynamic in a complete network is deterministic since the probability of a transition from one state to another is either 1 or 0. Note also that the state space is finite, which means that the elements of absorbing states can only be absorbing states or cycles. It remains to prove the correspondence between fixed points of $G(x)$ and absorbing states.

$\Rightarrow$) If x^* is a fixed point of $G(x)$, then assign actions to players according to the following rule: assign to the conformists whose tipping values are smaller than x^* the action 1 while to those whose tipping values are greater than x^* the action 0; assign to the anti-conformists whose tipping values are smaller than x^* the action 0 while to those whose tipping values are greater than x^* the action 1. Obviously this action profile corresponds to one absorbing state since nobody would like to change actions next period.

$\Leftarrow$) If $x^* = (x_1, x_2, \dots, x_n)$ is an absorbing state, then $\overline{x} = {}^1\!/_n \sum_{i=1}^{p} x_i$ is a fixed point of $G(x)$. By contradiction, if $\overline{x} > G(\overline{x})$, there will be some players playing action 0 at the present period who would like to play action 1 in the next period (e.g. conformists $i \in N_c$ with $\mu_i < \overline{x}$ or anticonformists $j \in N_a$ with $\mu_j > \overline{x}$). It is similar for the case $\overline{x} < G(\overline{x})$. Thus $\overline{x} = G(\overline{x})$.

B Proof of Proposition 1

Note that $x \in \{0, {}^1\!/_n, \dots, {}^{n-1}\!/_n, 1\}$.

Fix $x \geq \mu_a$. All conformist agents with threshold less than or equal to x would like to take action "1" when observing x (with proportion x). The anticonformist agents with threshold μ_a as well as all conformist agents with threshold strictly greater than x would like to take action "0" when observing x. Thus $G(x) = x$.

Fix $x < \mu_a$. All conformist agents with threshold less than or equal to x as well as the anticonformist agents with threshold $\mu_a > x$ would like to take action "1" when observing x (with proportion $x + {}^2\!/_n$). All conformist agents with threshold strictly greater than x would like to take action "0" when observing x. Thus $G(x) = x + {}^2\!/_n > x$. As a conclusion, $\mathcal{F} = \{{}^k\!/_n, \dots, {}^{n-1}\!/_n\}$ is the set of fixed points of the function G. By Theorem 2, it is also the set of absorbing states of the opinion dynamics. Obviously, $|\mathcal{F}| = n(1 - \mu_a) = n - k$.

Starting from any initial state with the group opinion $x^* = {}^{k^*}\!/_n$, if $xk^* \geq k$ and $k^* \neq n$, then $k^* \in \mathcal{F}$. Thus x^* is a reachable absorbing state. If $k^* < k$, then $G(x) = x + {}^2\!/_n$. It means that two more conformist agents will be activated at the next stage. This "domino" effect stops till $x \geq \mu_a$ with $x = {}^k\!/_n$ or $x = {}^{k+1}\!/_n$ depending on which one has the same parity as k^*.

C Proof of Proposition 2

Fix $x \in \{0, {}^1\!/_n, \dots, {}^{n-1}\!/_n, 1\}$. If $x < \mu_a^1$, only the conformist agents with threshold less than or equal to x as well as the two anticonformist agents would take action "1" at the next stage (with proportion $x + {}^3\!/_n$ in total). Thus $G(x) = x + {}^3\!/_n > x$. Similarly, if $x \in [\mu_a^1, \mu_a^2[$, $G(x) = x + {}^1\!/_n > x$; if $x \in [\mu_a^2, 1[$, $G(x) = x - {}^1\!/_n < x$.

Obviously, $G(1) = {}^{n-2}\!/_n < 1$. Therefore, $\forall x \in S$, $G(x) \neq x$. By Theorem 2, there is no absorbing state.

To show that this dynamic end up with a cycle regardless of the initial state, let us distinguish the following cases.

Assume that the dynamic start with the state $x = {}^{k_2}\!/_n$. Then all conformist agents with threshold less than x (with proportion ${}^{k_2-1}\!/_n$) would take action "1". All conformist agents with threshold greater than x as well as the two anticonformist agents would take action "0" at the next stage (v_2 with $x = {}^{k_2-1}\!/_n$). Then, observing $x < \mu_a^2$, the anticonformist agents with threshold μ_a^2 would change her action into "1" at the following stage (v_1 with $x = {}^{k_2}\!/_n$). Similar analysis is applied to the initial state $x = {}^{k_2-1}\!/_n$.

Assume the initial state satisfies $x < {}^{k_2-1}\!/_n$. If $x < \mu_a^1$, then $G(x) = x + {}^3\!/_n$. After every stage, there will be 3 more types of conformist agents would like to take action "1". This activation process stops till $x \in [\mu_a^1, \mu_a^2]$, then $G(x) = x + {}^1\!/_n$. After every stage, there will be one more types of conformist agents would like to take action "1". This activation process stops till $x = \mu_a^2 = {}^{k_2}\!/_n$. Then it goes back to the first case and forms a cycle $v_2 \to v_1 \to v_2$.

Assume the initial state satisfies $x \in [{}^{k_2}\!/_n, 1[$, then $G(x) = x - {}^1\!/_n$. This desactivation process stops till $x = {}^{k_2}\!/_n$. Then it goes back to the first case and forms a cycle $v_2 \to v_1 \to v_2$.

D Proof of Theorem 3

Fix $x \in \{0, {}^1\!/_n, \dots, {}^{n-1}\!/_n, 1\}$.

(i) If $k_1 \neq 0$, and if $s \in [0, {}^{k_1}\!/_n[$, $G(x) = x + {}^{2\ell+2}\!/_n > s$. If $x \in [{}^{k_{2\ell+1}}\!/_n, 1[$, $G(x) = x - {}^{2\ell}\!/_n < x$.

In general, if $x \in [{}^{k_i}\!/_n, {}^{k_{i+1}}\!/_n[$ $(i = 1, \dots, 2\ell)$, only the anti-conformist agents with threshold strictly greater than ${}^{k_i}\!/_n$ (with proportion ${}^{2\ell+1-i}\!/_n$) and the conformist agents with threshold less than or equal to x (with proportion $x + {}^1\!/_n - {}^i\!/_n$) would like to take action 1. That is, $G(x) = x + {}^{2\ell+2-2i}\!/_n$.

By $G(x) = x$, we get $i = k + 1$. Therefore, $\forall x \in [{}^{k_{k+1}}\!/_n, {}^{k_{k+2}}\!/_n[$, $G(x) = x$. Thus the set of fixed points of $G(x)$ is $\mathcal{F} = \{{}^{k_{k+1}}\!/_n, \dots, {}^{k_{k+2}-1}\!/_n\}$. By Theorem 2, the absorbing states are the action profiles associated to $\mathcal{F}$.

(ii) If $k_1 \neq 0$, and if $x \in [0, {}^{k_1}\!/\!_n[$, $G(x) = x + {}^{2\ell+1}\!/\!_n > s$. If $x \in [{}^{k_{2\ell}}\!/\!_n, 1[$, $G(x) = x - {}^{2\ell-1}\!/\!_n < x$.

In general, if $x \in [{}^{k_i}\!/\!_n, {}^{k_{i+1}}\!/\!_n[$ $(i = 1, \dots, 2\ell - 1)$, only the anti-conformist agents with threshold strictly greater than ${}^{k_i}\!/\!_n$ (with proportion ${}^{2\ell-i}\!/\!_n$) and the conformist agents with threshold less than or equal to x (with proportion $x + {}^1\!/\!_n - {}^i\!/\!_n$) would like to take action 1. That is, $G(x) = x + {}^{2\ell+1-2i}\!/\!_n$.

By $G(x) = x$, we get $i = k + {}^1\!/\!_2$. But i should be an integer. Thus $G(x)$ has no fixed point (i.e., $\mathcal{F} = \emptyset$). By Theorem 2, there is no absorbing state but cycles.

It remains to prove the statement on cycles. For this, we observe the following property of the function $G(x)$: when x changes from ${}^i\!/\!_n$ to ${}^{(i+1)}\!/\!_n$, the value of G is increased or decreased by one unit, depending whether agent i is conformist or anti-conformist. Hence for G the variation in ordinate cannot be greater than the variation in abscissa. We proceed in three steps.

1. The sequence of points $(x, (Gx)), (y, G(y)), (x, G(x))$ is a cycle iff it corresponds to a sequence of transitions, i.e., $y = G(x)$, $x = G(y) = G^{(2)}(x)$.

2. We show that a cycle of length 3 cannot exist. Let $(x, G(x)) \to (G(x), G^{(2)}(x)) \to (G^{(2)}(x), G^{(3)}(x)) \to (x, G(x))$ be a cycle. This implies $G^{(3)}(x) = x$. As the origin of the cycle is unimportant, suppose that $(x, G(x))$ is the leftmost point, i.e., $x < \min(G(x), G^{(2)}(x))$. Observe that this entails that this point is above the diagonal $(x < G(x))$. We may suppose that the second point $(G(x), G^{(2)}(x))$ is also above the diagonal, so that we have $x < G(x) < G^{(2)}(x)$, which entails that the 3d point is below the diagonal since its ordinate is x. By the above observation on the function G, the jump in ordinate cannot exceed the jump in abscissa. In particular, concerning the jump between the 2nd and 3d point, we obtain $G^{(2)}(x) - x \leq G^{(2)}(x) - G(x)$, which cannot be true as $x < G(x)$. Suppose now that the second point is below the diagonal. As the 1st point is the leftmost point, we necessarily have $x < G^{(2)}(x) < G(x)$. The condition on the jump between the 1st point and the 3d point yields $G(x) - x \leq G^{(2)}(x) - x$, a contradiction with $G^{(2)}(x) < G(x)$. The case where the first point is under the diagonal works similarly.

3. We show that no cycle of length greater than 3 can exist. Simply observe that selecting the leftmost point and 2 other points in a sequence of more than 3 points amounts to the case of a cycle of length 3 as the jump conditions will impose the same contradictions.

E Proof of Proposition 4

(1) Consider the case that $\nexists i \in N$ s.t. $\mu_i = \mu_a$.

$\Leftarrow$ **(Sufficiency)** Fix $x \in \{0, {}^1\!/\!_n, \cdots, {}^{n-1}\!/\!_n, 1\}$.

If $x \in [0, \mu_1[$, only anti-conformist agents would like to take action 1. That is, $G(x) = \delta_a$. But by the first inequality of 8, $G(x) > x$.

If $x \in [\mu_{i_0}, \mu_{i_0+1}[$, $(i_0 = 1, \dots, k-1)$, only anti-conformist agents and the conformists agents with threshold less than or equal to μ_{i_0} would like to take action 1. That is, $G(x) = \delta_a + \sum_{i=1}^{i_0} q_i$. By the second inequality of 8, $G(x) > x$.

Similarly, if $x \in [\mu_k, \mu_a[$, $G(x) = \delta_a + \sum_{i=1}^{k} q_i > x$; if $x \in [\mu_a, \mu_{k+1}[$, $G(x) = \sum_{i=1}^{k} q_i < x$; if $x \in [\mu_{i_0}, \mu_{i_0+1}[$, $(i_0 = k+1, \ldots, p-1)$, $G(x) = \sum_{i=1}^{i_0} q_i < x$; if $x \in [\mu_p, 1[$, $G(x) = \sum_{i=1}^{p} q_i < x$.

As a conclusion, there is no absorbing state.

$\Rightarrow$ **(Necessity)** We provide a *reductio ad absurdum* proof. Suppose the thresholds and corresponding fractions do not satisfy inequalities 8. Then distinguish the following cases.

- $\delta_a < \mu_1$

  In this case, $x = \delta_a$ is a fixed point of $G(x)$ since given the group opinion δ_a, only the anti-conformist agent with threshold strictly greater than μ_1 would take action 1 (with proportion δ_a). Then the group opinion at the next stage is still δ_a.

- $\exists$ some $i_0 \in \{1, 2, \cdots, k-1\}$, such that $\delta_a + \sum_{i=1}^{i_0} q_i < \mu_{i_0+1}$.

  Assume that i_0^* is the smallest number satisfying this condition. That is, $\delta_a + \sum_{i=1}^{i_0^*} q_i < \mu_{i_0^*+1}$ and $\forall i_0 < i_0^*$, $\delta_a + \sum_{i=1}^{i_0} q_i \geq \mu_{i_0+1}$. Thus $\delta_a + \sum_{i=1}^{i_0^*} q_i < \mu_{i_0^*+1} < \mu_a < \mu_{k+1} < \cdots < \mu_p$ and $\mu_1 < \cdots < \mu_{i_o^*} < \delta_a + \sum_{i=1}^{i_0^*} q_i$. In this case, given the group opinion $s = \delta_a + \sum_{i=1}^{i_0^*} q_i$, only the conformist agent $i \leq i_0^*$ as well as the anti-conformist agent will take action "1" (with proportion $\sum_{i=1}^{i_0^*} q_i + \delta_a$ in total) at the next stage. So $\delta_a + \sum_{i=1}^{i_0^*} q_i$ is a fixed point of $G(x)$.

- $\delta_a + \sum_{i=1}^{k} q_i < \mu_a$

  Then $\delta_a + \sum_{i=1}^{k} q_i < \mu_a < \mu_{k+1} < \cdots < \mu_p$. On the other hand, $\forall i_0 = 1, \ldots, k-1$, $\delta_a + \sum_{i=1}^{i_0} q_i \geq \mu_{i_0+1}$ (Otherwise it coincides with the previous case), then $\forall i_0 = 1, \ldots, k-1$, $\mu_{i_0+1} \leq \delta_a + \sum_{i=1}^{i_0} q_i < \delta_a + \sum_{i=1}^{k} q_i$ (since $i_0 < k$). In this case, given the group opinion $x = \delta_a + \sum_{i=1}^{k} q_i$, only the anti-conformist agent and the conformist agent $i = 1, \ldots, k$ will take action 1 at the next stage with proportion $\delta_a + \sum_{i=1}^{k} q_i$ in total. Thus $\delta_a + \sum_{i=1}^{k} q_i$ is a fixed point of $G(x)$.

- $\exists$ some $i_0 \in \{k+1, k+2, \cdots, p\}$, such that $\sum_{i=1}^{i_0} q_i \geq \mu_{i_0}$.

  Assume that i_0^{**} is the largest number satisfying this condition. That is, $\sum_{i=1}^{i_0^{**}} q_i \geq \mu_{i_0^{**}}$ and $\forall i_0 > i_0^{**}$, $\sum_{i=1}^{i_0} q_i < \mu_{i_0}$. Thus $\sum_{i=1}^{i_0^{**}} q_i \geq \mu_{i_0^{**}} > \mu_{i_0^{**}-1} > \cdots > \mu_a > \cdots > \mu_1$ and $\forall i_0 > i_0^{**}$, $\sum_{i=1}^{i_0^{**}} q_i < \sum_{i=1}^{i_0} q_i < \mu_{i_0}$. In this case, given the group opinion $x = \sum_{i=1}^{i_0^{**}} q_i$, only the conformist agent $i < i_0^{**}$ will take the action 1 at the next stage with proportion $\sum_{i=1}^{i_0^{**}} q_i$ in total. So $\sum_{i=1}^{i_0^{**}} q_i$ is a fixed point of $G(x)$.

- $\sum_{i=1}^{k} q_i \geq \mu_a$

  Then $\sum_{i=1}^{k} q_i \geq \mu_a > \mu_k > \mu_{k-1} > \cdots, \mu_1$. On the other hand, $\forall i_0 \in \{k+1, k+2, \cdots, p\}$, $\mu_{i_0} > \sum_{i=1}^{i_0} q_i > \sum_{i=1}^{k} q_i$. In this case, given the group opinion $x = \sum_{i=1}^{k} q_i$, only the conformist agent $i = 1, \ldots, k$ will take action 1 at the next stage with proportion $\sum_{i=1}^{k} q_i$. Thus $s = \sum_{i=1}^{k} q_i$ is a fixed point of $G(x)$.

As a conclusion, the fixed point of $G(x)$ always exists. By Theorem 2, the absorbing state always exists which leads to a contradiction.

(2) It is analogous for the case that $\exists i \in N$ s.t. $\mu_i = \mu_a$.

F Proof of Theorem 4

Suppose that there is no absorbing state, hence we are under the conditions of the system of inequalities (8) or (9). It remains to prove the statement on the cycle. The transition function G has the following behavior: considering that x grows from 0 to 1, it starts from the value $G(0) = \delta_a$, then increases at each point $x = \mu_i$ by the quantity q_i, and decreases by the quantity δ_a at point μ_a. Therefore, $G(\mu_k) = \delta_a + \sum_{i=1}^k q_i$ and $G(\mu_a) = \sum_{i=1}^k q_i$. Then $G(x)$ continues to increase at each value μ_i when x goes from μ_a to 1. It follows that the inequalities (8) (or (9)) imply the following properties of the transition function:

(i) The three first inequalities imply that $G(x) > x$ for all $x < \mu_a$. Hence the part of the transition function to the left of μ_a is strictly above the diagonal and nondecreasing;

(ii) The 4th inequality implies that $G(\mu_a) < \mu_a$;

(iii) The last inequality implies that $G(x) < x$ for all $x > \mu_a$, hence the part of the transition function to the right of μ_a including this point is strictly below the diagonal and nondecreasing.

We consider the square delimited by the diagonal points $(\sum_{i=1}^k q_i, \sum_{i=1}^k q_i)$ and $(\sum_{i=1}^k q_i + \delta_a, \sum_{i=1}^k q_i + \delta_a)$. We claim that if any point $(x, G(x))$ is chosen inside this square, the "next" point $(G(x), G(G(x)))$ is still inside the square. First observe that $\sum_{i=1}^k q_i$ and $\sum_{i=1}^k q_i + \delta_a$ are respectively the minimum value and the maximum value achieved by G in the interval $[\sum_{i=1}^k q_i, \sum_{i=1}^k q_i + \delta_a]$, hence if x lies in this interval, $(x, G(x))$ is in the square. Now, taking x to the right of μ_a, we have that $\sum_{i=1}^k q_i \leq G(x) < x \leq \sum_{i=1}^k q_i + \delta_a$, so by the previous observation $(G(x), G(G(x)))$ lies in the square. Similarly, if x is on the left of μ_a, then $\sum_{i=1}^k q_i \leq x < G(x) \leq \sum_{i=1}^k q_i + \delta_a$, so that again the image of the point by G is still in the square.

The number of different values (levels) of G in the square is the number of values μ_i in the interval $\left]\sum_{i=1}^k q_i, \sum_{i=1}^k q_i + \delta_a\right]$ plus one (taking into account right-continuity) and plus one corresponding to μ_a. As this number is finite, the successive points $(x, G(x))$, $(G(x), G(G(x))), \ldots$ must from some step form a cycle.

We prove the claim on the upper bound. The number of levels of G is maximal when the gap q_i at each μ_i is minimal. The minimal value of q_i is $1/n$, which yields a total number of levels to be $n\delta_a + 1$ because the total gap is δ_a.

G Proof of Theorem 5

1. Suppose that the state is T with $t \geq d$ and $n - t \geq d$. This is possible because by the assumptions $n_a \geq d$, $n_c \geq d$, we have $d \leq n/2$. We claim that a transition to any $Q \in 2^N$ is possible. Indeed, we can have any type of neighborhood, so that the average opinion $\overline{a}(T)$ in a neighborhood can be any value in $\{0, 1/d, \ldots, 1\}$. It follows that $p_i^0(T)$ and $p_i^1(T)$ can be positive for all conformists and all anti-conformists.

2. Consider that either $t < d$ or $n - t < d$. It suffices to prove that a transition to some set Q such that $q \geq d$ and $n - q \geq d$ is possible to conclude the proof. Suppose $t < d$ (the other case is similar). Then $n - t > d$, so that the 0-neighborhood has a positive probability. Supposing that all players take the 0-neighborhood, the next action will be 0 for the conformists and 1 for the anti-conformists. Hence $Q = N_a$, which does the job as $|N_a| \geq d$ and $|N \setminus N_a| = |N_c| \geq d$.

H Proof of Theorem 6

Recall that at each time step, a random neighborhood of random size is drawn, for each agent. Each agent has a different threshold but it is fixed (distribution is known)

1. Consider a state T. In order to have for every conformist and anti-conformist a possibility of choosing action 1 and 0, we must have for conformist choosing action 1 $P(\overline{a}_i \geq \mu_i : S; d) > 0$ for at least one d and choosing action 0 $P(\overline{a}_i \geq \mu_i : S; d) < 1$ for at least one d in the support. For action 1 we must have T with $t \geq \overline{\mu}\underline{d}$ (must work for all agents) and for action 0 we must have $N \setminus T$ with $n - t > \underline{d}(1 - \underline{\mu})$. The conditions are inverted for anti-conformists. Then a transition to any Q is possible at next step.

2. Suppose now that T is such that $t < \overline{\mu}\underline{d}$. Suppose $n_a \geq \overline{\mu}\underline{d}$ and $n_a < n - \underline{d}(1 - \underline{\mu})$. (same conditions for n_c)(hence equivalently, $n_a > \underline{d}(1 - \underline{\mu})$ and idem for n_c). It follows that $\overline{\mu}\underline{d} \leq n/2$ and $\underline{d}(1 - \underline{\mu}) < n/2$. Observe that we have then

$$t < \overline{\mu}\underline{d} \leq {}^{n}/_{2} < n - \underline{d}(1 - \underline{\mu}).$$

Therefore every conformist can choose action 0 and every anti-conformist can choose action 1, so that a transition to N_a is possible. As by assumption $n_a \geq \overline{\mu}\underline{d}$ and $n_a < n - \underline{d}(1 - \underline{\mu})$, we are back to Step 1 and a transition to any state Q is possible.

The case where $t \geq n - \underline{d}(1 - \underline{\mu})$ works similarly as we have

$$t \geq n - \underline{d}(1 - \underline{\mu}) \geq {}^{n}/_{2} \geq \overline{\mu}\underline{d}.$$

Then a transition to N_c is possible, which allows then a transition to any state Q.