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Abstract

This paper incorporates endogenously emerging beliefs, with an emphasis on the role of social influence, into a stylized Islands model, characterized by uncertainty and strategic complementarity generating frictional coordination. In particular, individuals can have pessimistic, neutral or optimistic beliefs and they can change these beliefs over time following a switching mechanism, driven both by economic outcomes and social influence. In such a framework, we study the emergent dynamics in order to assess the impact social influence has on agents' coordination, economic stability and welfare. We find that in the absence of social influence rational expectations are unstable and agents coordinate over time on a pessimistic and highly inefficient stationary state in which output and welfare are below the rational expectations equilibrium. As the importance of social influence grows, the steady state becomes even more pessimistic. As it crosses a certain threshold, additional equilibria emerge and the economy may converge to the rational expectations steady state, in which welfare is highest, or to a much more optimistic equilibrium, which is however not necessarily more efficient. We conclude that social influence can act as a coordination device able to smooth the impact of uncertainty and individual incentives, with positive effects on welfare.

JEL codes: D83, D84, E10, E71.
Keywords: animal spirits, heterogeneous expectations, social influence, uncertainty.

1 Introduction

The role played by psychological factors in macroeconomics has been recently gaining increasing attention. Within general equilibrium literature, for instance, there is a growing

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number of models which incorporate the notion of animal spirits, i.e. waves of optimism and pessimism[1], as exogenous shocks to beliefs or confidence and show that they are able to produce economic fluctuations[2,3].

In such models, and more generally in macroeconomics, however, the notion that individuals' beliefs formation critically depends also on their interaction with each other, which is familiar in several fields, is not as much acknowledged. This hampers a thorough understanding of the macroeconomic implications of such interaction on several aspects such as output and welfare[4]. In fact, works in psychology and sociology have carefully examined the role of social influence and its drivers, assessing its impact on individuals' opinion formation. For instance, previous studies have identified both informational and normative motivations to conform with others. The former, reinforced under uncertainty, are based on the goal to form a correct interpretation of reality; the latter deal with the desire of social approval. Moreover, perceived consensus matters: all else being equal, an individual will be more likely to adapt to the opinions and behavior displayed by the (local) majority than by the (local) minority[5].

As regards to general equilibrium models, one of the reasons for the limited consideration of social influence may partly lie in the their focus on the consequences of beliefs or confidence shock rather than on its endogenous emergence, about which the modelers are agnostic.

The present paper builds on such considerations and proposes a way to formalize endogenously emerging beliefs, i.e. optimism and pessimism, with an emphasis on the role of social influence, into a stylized Islands model, similar to that developed by Angeletos and La'O (2013). Such a model is purposely simple in order to maintain the reasoning as close as possible to that of standard neoclassical demand and supply functions. Its crucial features are uncertainty and strategic complementarity. In particular, producers are not aware of the demand they will face on the market and an increase in one's output acts as an incentive for others to raise production as well, and the other way round. Combined, these two features generate frictional coordination and a hierarchy of higher-order beliefs, which reminds a Beauty Contest type of game[6] that may lead individuals to lower their

[1]In Keynes' definition, animal spirits are "a spontaneous urge to action rather than inaction", thus, they specifically correspond to an enterprising or optimistic state, while the absence of animal spirits would resemble pessimism. However, in most recent literature, animal spirits is used to indicate waves of optimism or pessimism.

[2]Cfr., among others, Milani (2011), Angeletos and La'O (2013), Benhabib, Wang, and Wen (2015), Huo and Takayama (2015), Acharya, Benhabib, and Huo (2017), Angeletos, Collard, and Dellas (2018).

[3]Moreover, there are heterogeneous agents models deviating from the rational expectations hypothesis which study the dynamics among beliefs and economic outcomes. Cfr. e.g. Hommes (2006); Franke and Westerhoff (2017).

[4]Few exceptions among economists are, for example, Keynes (1936), Shiller, Fischer, and Friedman (1984), Kindleberger (1989).

[5]Cfr. Cialdini and Goldstein (2004), Flache, Mäs, Feliciani, Chattoe-Brown, Deffuant, Huet, and Lorenz (2017).

[6]The reference to such a game can be found in Keynes (1936).

production well below the efficient level, hampering overall welfare. Agents may not be able to coordinate well their beliefs and behavior, eventually driving the economy to a bad equilibrium[7].

This framework is developed by Angeletos and La'O (2013) to show that exogenously induced sentiment shocks can lead to aggregate demand-driven fluctuations. However, the main features of the model strengthen the relevance of studying endogenous beliefs and social influence. Uncertainty, in fact, might create incentives for people to observe and imitate others' opinion, while the trade off between social and individual objectives can be strongly shaped by interaction among agents. Moreover, such interesting aspects are brought to their deeper implications on stability and efficiency if combined with endogenously emerging beliefs.

Within this framework, we study the emerging sentiment dynamics and we assess the impact of social influence on agents' coordination, stability and efficiency of our simple trade economy. In particular, we pose the following questions. First, in the absence of social influence, does the economy converge to rational expectations (RE)? This is an important theoretical issue, because on the one hand it sheds light on the consistency conditions of the RE assumption, on the other it questions whether rational expectations necessarily generate higher profits for the individuals than other forecasting rules. Secondly, we study how optimism and pessimism endogenously evolve over time. Third, we assess the effects of social influence on the sentiment dynamics: does it bring about new steady states, or modify the existing ones? Also, what is the impact of social influence on profits, production and welfare? Finally, to sum up the different insights, we explore the role of social influence as a coordination device, i.e. a mechanism that can improve agents' coordination and enhance economy's welfare, through its impact on sentiment dynamics.

More in detail, the economy is composed of islands that trade in every period in random pairs; at the moment in which they take their production and employment decisions, islands do not know the island they have been matched with, along with her productivity and beliefs. Therefore, islands need to form expectations on the trading partner's output level and on her higher-order beliefs. In this respect, islands can have optimistic, pessimistic or neutral expectations and they can switch their type over time. The switching mechanism, similar to that developed by Lux (1995), depends on economic outcomes, i.e. the difference between the expected profits earned by each expectation type, and on social influence, that is, the observation and, eventually, the imitation of others' sentiment. We assume that both of them are public knowledge: this might be interpreted as due to news media which, every period, publicly announce the value of expected profits and the prevalent attitudes in the population.

We explore the system considering an economy populated by neutral and pessimistic

[7] Lack of coordination may consist of the presence of multiple equilibria, as in the literature on sunspots (e.g. Cass and Shell (1983) and Azariadis (1981)), or in the coordination towards a bad equilibrium.

islands alone, and one which involves optimistic agents, too. We find that when all islands are neutral, their expectations are rational. Moreover, in the absence of social influence agents coordinate over time on a pessimistic and highly inefficient stationary state in which output is below the RE equilibrium; hence, RE are unstable. Agents' beliefs on the trading partner's output determine their amount of production which influences negatively their terms of trade and positively their total costs; therefore, individuals have incentives to switch from the neutral belief to the pessimistic one in order to improve their terms of trade and sustain lower expenditures. As the importance of social influence in the individuals' belief switching process grows, this undesirable stable steady state becomes even more pessimistic, but as the social influence parameter crosses a certain threshold a second stable steady state emerges, in which agents coordinate on a much less pessimistic belief, eventually converging to the most efficient outcome. The intuition is that whereas lower levels of social influence reinforce the individuals' incentives towards the pessimistic belief, a stronger impact of social influence is able to counteract those incentives by pushing agents to imitate other expectation types as well. In the economy populated by all three types of beliefs, a higher impact of social influence generates also a new optimistic stable equilibrium, characterized by higher production but not necessarily higher welfare. Hence, social influence has a strong impact on the sentiment dynamics; extreme levels generate equal basins of attractions to those stable fixed points characterized by the entire predominance of either pessimistic, neutral or optimistic islands. In addition, social influence is able to balance the incentives to lower further and further output with respect to the trading partner. Concerning welfare, we find that the optimal outcome occurs under rational expectations, i.e. when all islands are neutral. This result points out to the presence of a trade off between individual incentives and social outcomes, which social influence can resolve by helping agents coordinate on better stable steady states and neutralizing frictional coordination. However, social influence does not necessarily have a monotone effect on welfare: it can worsen the pessimistic equilibrium and, under certain conditions, an increase of the percentage of neutral or optimistic islands may bring about a reduction in welfare.

The rest of the paper is organized as follows. The next section reviews the relevant literature; section 2 presents the basic structure of the model on which we incorporate endogenous beliefs along with a switching mechanism, discussed in section 3. Section 4 analyzes the sentiment dynamics, without and with social influence; section 5 assesses its impact on output and welfare. Section 6 concludes. Appendix A provides the derivation of some formulas presented in the paper; Appendix B includes the proofs of the propositions and in Appendix C the robustness of results with respect to parameter variations is studied.

1.1 Literature Review

Macroeconomic models incorporating psychological factors are growing in number and importance; regardless of the specific formalization, the notion that individuals may be not entirely rational is often taken into account. This idea finds strong support from other disciplines as well, such as, e.g., psychoanalysis, psychology, neuroscience and sociology. In these fields over time an extensive literature has developed on how individuals make decisions by integrating their cognitive abilities with their emotional state. Furthermore, in a context of uncertainty and complexity, agents might rely on simple behavioral rules, rather than complex mathematical reasoning (cfr., among others, in psychoanalysis and psychology: Simon (1990), Gigerenzer, Todd, ABC Research Group, et al. (1999), Tuckett (2012), Lerner, Li, Valdesolo, and Kassam (2015); in neuroscience: Bechara, Damasio, and Damasio (2000); in sociology: Smelser (1998)).

In macroeconomics, several theories have been proposed to incorporate such insights, some focusing on the consequences of sentiment shocks, while others on their endogenous emergence. The present paper is related to them in multiple ways exposed below.

Within the former group, in the general equilibrium literature there is a number of works that develop the notion of fluctuations driven by non-fundamental uncertainty. For instance, Angeletos and La'O (2013), Huo and Takayama (2015), Acharya, Benhabib, and Huo (2017), Angeletos, Collard, and Dellas (2018) formalize exogenous shocks on non-fundamental higher-order beliefs in unique equilibrium models which give a central position to frictional coordination, by including strategic complementarity and lack of common knowledge. We relate to this literature by incorporating such insights on coordination issues in our model. Moreover, Benhabib, Wang, and Wen (2015) show that in standard economies – without informational frictions, externalities, non-convexities or strategic complementarities in production – optimal decisions based on sentiments can generate stochastic self-fulfilling RE equilibria. Such works on extraneous uncertainty are rooted in a literature developed in the 1980s on multiple equilibria and sunspots fluctuations, which aimed at reintroduce the Keynesian narrative on animal spirits and multiple equilibria in standard RBC models (cfr. e.g. Azariadis (1981), Cass and Shell (1983), Cooper and John (1988), Benhabib and Farmer (1994)). Mentioned papers, in general, assume rational expectations, following the long tradition initiated by the seminal contribution of Muth (1961). An exception is provided by Milani (2011), which assumes that agents learn model coefficients over time and the learning process is affected by expectations shocks.

Endogenously emerging beliefs and heterogeneous expectations have also been discussed in the economics literature. One early example applied to financial markets is the Santa-Fe Artificial Stock Market (Arthur, Holland, LeBaron, Palmer, and Tayler, 1996), in which traders every period select an expectation rule among a large pool of rules, based on the market conditions that they observe and on a fitness measure of the rules. A simpler approach to formalize heterogeneous expectations is provided by the Brock-Hommes model (Brock and Hommes, 1997, 1998) in which there is a finite set of simple forecasting rules among which agents can choose – e.g. naive and rational expectations. Moreover, they introduce a switching mechanism among forecasting rules, which depends on the rules'(relative) performance measure. Another approach within the field of heterogeneous expectations, known as adaptive learning or statistical learning (e.g. Evans and Honkapohja (2012)), considers agents as using the perceived law of motion of the economy as a forecasting rule and trying to learn the optimal parameters with some learning mechanism, as new realizations become available (e.g. ordinary least squares, OLS, sample autocorrelation).

In the following papers, instead, the heterogeneous expectations approach has been used to explicitly model animal spirits, as waves of optimism and pessimism. Franke and Westerhoff (2017) provide an overview of some of these models. Examples are De Grauwe (2011), in which, as in the present paper, agents can be optimistic or pessimistic about future output and inflation with respect to the rational expectations benchmark. In Anufriev, Assenza, Hommes, and Massaro (2013), the authors implement the heterogeneous expectations framework of Brock and Hommes (1997) in a frictionless DSGE model to study the role of heterogeneous expectations about future inflation. They consider a simple case in which agents can choose among three expectation rules – one fundamentalist, one with a positive bias and one with a negative bias.

Furthermore, the present work is related to the following papers which consider the role of social influence. In particular, our switching mechanism is grounded on that introduced by Lux (1995), which formalizes a financial market where booms and busts are driven by the change in the number of optimistic and pessimistic investors, explicitly driven by social influence. In particular, there are chartist and fundamentalist investors and the former group is composed by optimists and pessimists. The probability of an optimist becoming pessimistic and *vice versa* depends on the difference in the shares of the two types of beliefs. In a distinct framework, Burnside, Eichenbaum, and Rebelo (2016) propose a different way to explore the impact of social influence on booms and busts in the housing markets. They consider three types of agents which can 'infect' each other with their forecasting rule, depending on their confidence level.

To sum up, we try to combine insights from the general equilibrium literature focusing on coordination issues, with contributions from the literature on endogenously emerging beliefs, in particular that on animal spirits. Furthermore, we relate to those studies that incorporate the notion of social influence in order to examine sentiment dynamics and its effect on economic dynamics. Our work proposes new lens through which study endogenous beliefs and social influence in macroeconomics, by focusing on the hampering effects of frictional coordination and conflicting incentives existing in our economy. We examine whether social influence, in a context of uncertainty, can smooth these mechanisms and help agents coordinate on more efficient steady states. Thus, our contribution to the literature is twofold: first, we incorporate endogenously emerging beliefs in a stylized macro model which sheds light on uncertainty and frictional coordination; secondly, we look at social influence from this original perspective in order to assess its deep implications on several aspects of the economy.

2 The Model

The model describes an economy composed by a large number n of heterogeneous islands with different productivities. Similarly to Angeletos and La'O (2013), each island is inhabited by a locally owned firm and a single household. The former produces one good employing labor and land; the latter earns a wage by supplying labor to the local firm. Households do not save anything and they want to consume both the local good and the 'foreign' goods: this gives rise to trade among islands. Trade takes place in pairs through random matching: each period, every island is randomly matched with another one and it trades only with the selected partner[8]. Strategic complementarity arises because of the positive relationship between producers' output: higher supply from one island entails higher demand for another island. Or, seen through another lens, islands want to improve their terms of trade which are affected positively by the trading partner's output; however, they are also negatively influenced by one's own production, thus bringing about an incentive not to increase too much production with respect to the trading partner's one.

Timing of events is key. When islands take their production decisions, they are not aware of the island they have been matched with, along with her productivity and output. In particular, suppose that every period t unfolds in two different sub-periods: the 'morning' and the 'afternoon'. In the morning the random trading pairs are drawn and islands take their production and employment decisions, prior to observing their exact match and the terms of trade. Therefore, supply is determined under incomplete information about demand and islands need to form expectations on the trading partner's output level. In the afternoon islands actually meet and trade their previously determined output: households choose their consumption level of the 'home' good and the 'foreign' good and market-clearing prices are determined. Crucially, production and costs are decided upon in the morning based on beliefs, while revenues are obtained in the afternoon when information is complete.

Our model presents a simple economy, in which firms choose optimally the level of profits-maximizing production, households choose their consumption maximizing their utility and, finally, market-clearing prices equate the marginal utility of the two goods.

[8] We can interpret the home consumers as either being indifferent among the goods of all other islands, or as liking only the good of their current random match.

2.1 Households' consumption on island i

The household on island i maximizes the following utility function:

$$U_i = \Sigma_{t=0}^{\infty} \beta^t [U(c_{it}, c_{it}^*) - V(l_{it})], \tag{2.1}$$

where $\beta \in (0,1)$ is the discount factor, $c_{it} \in \mathbb{R}_+$ and $c_{it}^* \in \mathbb{R}_+$ are the consumptions of, respectively, the 'home' good and the 'foreign' good in time t. $l_{it} \in \mathbb{R}_+$ is the labor supply and $V(l)$ is the disutility of labor. U and V are given by:

$$U(c, c^*) = \left(\frac{c}{1-\eta}\right)^{1-\eta} \left(\frac{c^*}{\eta}\right)^{\eta} \qquad \text{and} \qquad V(l) = \frac{l^{\varepsilon}}{\varepsilon}, \tag{2.2}$$

where $\eta \in (0,1)$ is the fraction of 'home' expenditure spent on the 'foreign' good and $\varepsilon > 1$ is the Frisch elasticity of labor supply. The period t budget constraint for the household's utility maximization on island i is the following:

$$p_{it} c_{it} + p_{it}^* c_{it}^* \leq w_{it} l_{it} + r_{it} K + \pi_{it}, \tag{2.3}$$

where p_{it} and p_{it}^* are the prices of the 'home' good and the 'foreign' good, respectively. w_{it} is the wage, r_{it} the rental rate of land and π_{it} profits.

The first order conditions of the utility maximization[9] are $U_{c_{it}} = p_{it}$ and $U_{c_{it}^*} = p_{it}^*$. Moreover, trade between islands has to satisfy the trade balance condition, i.e. imports and exports have to be equal: $p_{it}^* c_{it}^* = p_{it}(y_{it} - c_{it})$, where y_{it} is the production of island i. In addition, all the production must be consumed, i.e. the market clearing condition must be satisfied: $c_{it} + c_{jt}^* = y_{it}$, where c_{jt}^* is the import in island j of the good produced on i. Combining these conditions, together with their corresponding version for i's trading partner j, we obtain the following results:

$$c_{it} = (1-\eta) y_{it}, \qquad c_{it}^* = \eta y_{jt} \qquad \text{and} \qquad p_{it} = y_{it}^{-\eta} y_{jt}^{\eta}. \tag{2.4}$$

From the above we can observe at the individual level the source of the strategic complementarity in our model. In particular, as shown in equation (2.4), a rise in y_{jt} increases the import of the 'foreign' good c_{it}^*, which raises the 'home' good's marginal utility and, in turn, its price. An increase in the price of island i's good can also be interpreted as an improvement of island i's terms of trade. In fact: $\frac{p_{it}}{p_{jt}} = \frac{y_{jt}}{y_{it}} = p_{it}^{\frac{1}{\eta}}$, which is an increasing function of p_{it}.

[9] After normalizing the local nominal prices so that the Lagrange multiplier $\lambda_{it} = 1$. See Angeletos and La'O (2013) for the proofs of the results in the present section.

2.2 Production on island i

Firms take production and employment decisions in the first stage of every period, when they still do not know the trading partner with which they have been matched with.

Island i's firm produces the local good, employing labor and land, with the following technology:

$$y_{it} = A_i l_{it}^{\Theta} k_{it}^{1-\Theta}, \tag{2.5}$$

where A_i is the local total factor productivity, which is formalized as a continuous random variable lognormally distributed[10]: $A_i \sim \log N(0, \sigma_A^2)$, with $\sigma_A > 0$. l_{it} and k_{it} are the labor and land input, respectively, and $\Theta \in (0,1)$ parametrizes the income share of labor. All islands are endowed with a fixed amount of land K.

Firms choose l_{it}, k_{it}, w_{it} and r_{it} optimally. In the labor market, the equilibrium wage is the wage which equates the marginal disutility of working with the marginal revenues of labor for the firm:

$$V'(l_{it}) = w_{it} = \mathbb{E}_{it}[p_{it}]\Theta \frac{y_{it}}{l_{it}}, \tag{2.6}$$

where $\mathbb{E}_{it}[p_{it}]$ is the expectation of island i on price p_{it}, which we will discuss in details later. The resulting optimal amount of labor input is given by $l_{it}^* = (\mathbb{E}_{it}[p_{it}]\Theta y_{it})^{\frac{1}{\varepsilon}}$.

As regards land, in equilibrium $k_{it}^* = K$, i.e. firms employ the total amount of land disposable on each island; thus, we set $K = 1$.

By inserting the optimal amount of labor in (2.5) and recalling that $p_{it} = y_{it}^{-\eta} y_{jt}^{\eta}$ and, thus, $\mathbb{E}_{it}[p_{it}] = y_{it}^{-\eta}\, \mathbb{E}_{it}[y_{jt}^{\eta}]$, we obtain the optimal level of output for island i:

$$y_{it} = K_1^{\alpha} A_i^{\alpha} [\mathbb{E}_{it}(y_{jt}^{\eta})]^{\theta\alpha}, \tag{2.7}$$

where $K_1 \equiv (\Theta^{\theta})$, $\theta \equiv \frac{\Theta}{\varepsilon} \in (0,1)$ and $\alpha \equiv \frac{1}{1-\theta+\eta\theta}$.

Equation (2.7) expresses that the equilibrium output of a producer is an increasing function of her productivity and of her expectation about the trading partner's output. In fact, higher productivity lowers the cost of producing and a higher supply by trading partners translates to higher demand for one's own production (or, equivalently, better terms of trade). $\eta\theta\alpha \in (0,1)$ represents the degree of the strategic complementarity. Equation (2.7) might be interpreted as the best response function in a two-player game: the players are the islands within a match and their actions are the amount of production. Nevertheless, this is a macro model in which islands are infinitesimal price takers and the complementarity is an outcome of competitive market interactions.

[10] Whereas in the initial characterization of the model and in the section on the switching mechanism we describe the economy as being composed by a large number n of islands, here we approximate n to a continuum of islands, in order to simplify some computations in the next sections. For $n \to \infty$, the approximation error goes to zero.

3 Endogenous beliefs and switching mechanism

Production decisions depend on islands' expectations about their match, i.e. $\mathbb{E}_{it}(y_{jt}^{\eta})$.

This is where we want to capture the idea of 'animal spirits', that is, optimism and pessimism, being driven not only by real economic variables but also by social influence.

3.1 Three types of endogenous belief

We consider three types of beliefs: neutral, optimistic and pessimistic. Below we formally define the neutral one and we use it as a benchmark to define the others.

When forming their beliefs on y_{jt}^{η}, islands know that their trading partner j's output, like theirs, is given by $K_1^{\alpha} A_j^{\alpha} [\mathbb{E}_{jt}(y_{it}^{\eta})]^{\theta\alpha}$. Therefore, they take that into consideration and form their first-order belief:

$$\mathbb{E}_{it}(y_{jt}^{\eta}) = \mathbb{E}_{it}\left(K_1^{\alpha\eta} A_j^{\alpha\eta} [\mathbb{E}_{jt}(y_{it}^{\eta})]^{\theta\alpha\eta}\right). \tag{3.1}$$

We assume that K_1^{α} and $\mathbb{E}_{it}(A_j^{\alpha\eta})$ are common knowledge[11]; thus, the subjective part of the expectation is $\mathbb{E}_{it}[\mathbb{E}_{jt}(y_{it}^{\eta})]^{\theta\alpha\eta}$. However, islands know y_{it}, so they can do a step forward in their reasoning and substitute it in equation (3.1):

$$\mathbb{E}_{it}(y_{jt}^{\eta}) = K_1^{\alpha\eta(1+\alpha\eta\theta)}\, \mathbb{E}(A_j^{\alpha\eta})^{(1+\alpha\eta\theta)} \underline{\underline{\mathbb{E}_{it}\left(\mathbb{E}_{jt}([\mathbb{E}_{it}(y_{jt}^{\eta})]^{\theta\alpha\eta})^{\theta\alpha\eta}\right)}}. \tag{3.2}$$

The underlined term in equation (3.2) describes the hierarchy of beliefs of island i up to the third-order belief, that is, what island i believes that island j thinks of the first-order belief of i on j. While one could go on indefinitely in the hierarchy of beliefs, we stop it at the third-order to define the 'neutral belief benchmark'. We assume that $\mathbb{E}_{it}\left(\mathbb{E}_{jt}[\mathbb{E}_{it}[y_{jt}^{\eta}]]\right) = \mathbb{E}_{it}(y_{jt}^{\eta})$,[12] that is, a neutral island i believes that the trading partner j knows her own (i's) belief. It is not crucial at which order we stop the hierarchy of beliefs as the following reasoning would be unchanged. We can now derive the neutral belief by inserting the assumed third-order belief on the right hand side of equation (3.2) and solving for the belief:

$$\mathbb{E}_{n}(y_{jt}^{\eta}) = K_1^{\eta\gamma}\, \mathbb{E}(A_i^{\eta\alpha})^{\frac{\gamma}{\alpha}} \tag{3.3}$$

where the subscript n stands for neutral, $\gamma \equiv \frac{1}{1-\theta}$ and $\frac{\gamma}{\alpha} \in (1, \infty)$[13].

In other words, a neutral island believes that her trading partner knows her own belief and this makes the hierarchy of beliefs collapse to the true fundamental.

[11] Therefore: $\mathbb{E}_{it}(A_j^{\alpha\eta}) = \mathbb{E}(A_j^{\alpha\eta}) = e^{\frac{(\sigma\alpha\eta)^2}{2}}$. See section A.1 for its derivation.

[12] Since we are dealing with point beliefs and not beliefs' distribution, this assumption translates directly into $\mathbb{E}_{it}\left(\mathbb{E}_{jt}([\mathbb{E}_{it}(y_{jt}^{\eta})]^{\theta\alpha\eta})^{\theta\alpha\eta}\right) = \mathbb{E}_{it}(y_{jt}^{\eta})^{(\theta\alpha\eta)^2}$.

[13] $\frac{\gamma}{\alpha} \equiv \frac{1-\theta+\eta\theta}{1-\theta}$.

The optimistic and pessimistic beliefs are defined as follows:

$$\mathbb{E}_{it}(y_{jt}^{\eta}) = \begin{cases} \mathbb{E}_n(y_{jt}^{\eta})(1+\delta) & \text{if } i \text{ is optimistic}(i=o) \\ \mathbb{E}_n(y_{jt}^{\eta})(1-\delta) & \text{if } i \text{ is pessimistic}(i=p) \end{cases} \tag{3.4}$$

where $\delta \in (0,1)$ represents the degree of optimism or pessimism.

The idea behind the deviation of δ from the neutral belief is that an optimistic (pessimistic) island i thinks that the trading partner j overestimates (underestimates) her own belief. In other words, an optimistic island is an island which believes that her future trading partner will be optimistic, while a pessimistic island is an island that believes that she will meet a pessimistic island. In this context, therefore, optimism and pessimism characterize the belief about the type of the future trading partner; they are, in fact, third-order beliefs. The implication of an optimistic (pessimistic) belief is that the agent thinks that the trading partner will produce more (less) than the neutral benchmark, and thus, because of the strategic complementarity, will in turn produce more (less). In fact, we observe what follows.

Output for the three types of islands By substituting equation (3.4) into (2.7), we obtain the optimal output for the different types of island:

$$y_i = \begin{cases} K_1^{\gamma} A_i^{\alpha}\, \mathbb{E}[A_j^{\eta\alpha}]^{\theta\gamma} & \text{if } i \text{ is neutral } (i=n) \\ K_1^{\gamma} A_i^{\alpha}\, \mathbb{E}[A_j^{\eta\alpha}]^{\theta\gamma}(1+\delta)^{\theta\alpha} & \text{if } i \text{ is optimistic } (i=o) \\ K_1^{\gamma} A_i^{\alpha}\, \mathbb{E}[A_j^{\eta\alpha}]^{\theta\gamma}(1-\delta)^{\theta\alpha} & \text{if } i \text{ is pessimistic } (i=p) \end{cases} \tag{3.5}$$

We can observe that, also with regard to production, optimists' and pessimists' output are positive and negative deviations, respectively, from the output of a neutral island. The size of this deviation is $(1+\delta)^{\theta\alpha}$ for the optimists and $(1-\delta)^{\theta\alpha}$ for the pessimists, where $\theta\alpha$ is the weight given by an island to the expectation of the trading partner's deviation from the neutral benchmark.

It is interesting to note the following proposition about the neutral belief.

Proposition 1 *When every island in the economy is neutral, the neutral belief $\mathbb{E}_n(y_j^{\eta})$ corresponds to the expected output of neutral islands $\mathbb{E}(y_n^{\eta})$. Therefore, in this case, neutral islands have rational expectations.*[14]

$$\mathbb{E}_n(y_j^{\eta}) = K_1^{\eta\gamma}\, \mathbb{E}(A_i^{\eta\alpha})^{\frac{\gamma}{\alpha}} = K_1^{\eta\gamma}\, \mathbb{E}(A_i^{\eta\alpha})^{\eta\gamma\theta} \int_0^{+\infty} A_i^{\eta\alpha} dF(A_i) = \mathbb{E}(y_n^{\eta}), \tag{3.6}$$

where on the left hand side there is the neutral belief and on the right hand side there is the expected value of neutral islands' production[15].

[14] See section B.1 for a proof.

[15] The same proposition does not hold for the neutral belief on y_j, which, when all islands are neutral, does not equal $\mathbb{E}(y_n)$.

Therefore, all islands being neutral reminds an economy under complete information in which all agents share the same information about one another, there is no higher-order uncertainty and beliefs only depend on true fundamentals.

By contrast, an economy populated by different belief types, can be thought as being under incomplete information, in which islands face uncertainty about one another's beliefs and actions. As a result, coordination is imperfect and islands make mistakes in their evaluations of others' beliefs.

Proposition 1 does not hold for the other types of island; in fact, from equation (3.5) we obtain optimists' and pessimists' expected production[16]:

$$\mathbb{E}(y_i^\eta) = \begin{cases} \mathbb{E}(y_n^\eta)(1+\delta)^{\theta\alpha\eta} & \text{if } i \text{ is optimistic}(i=o) \\ \mathbb{E}(y_n^\eta)(1-\delta)^{\theta\alpha\eta} & \text{if } i \text{ is pessimistic}(i=p). \end{cases} \tag{3.7}$$

Given that $\theta\alpha\eta \in (0,1)$, $1+\delta > (1+\delta)^{\theta\alpha\eta}$, that is, optimistic islands systematically overestimate optimists' expected production. On the other hand, $1-\delta < (1-\delta)^{\theta\alpha\eta}$, that is, pessimistic islands systematically underestimate pessimists' expected production. The reason, therefore, lies in the degree of strategic complementarity.

Profits Profit function of an island i trading with an island j in period t is given by[17]

$$\pi_{ij} = y_i^{1-\eta}(y_j^\eta - \mathbb{E}_i(y_j^\eta)) \tag{3.8}$$

and thus depends on the type and productivity of i and j. Equation (3.8) is key to understand islands' motivations to lower their beliefs as it shows how uncertainty and strategic complementarity shape individual incentives. In particular, i's belief on the trading partner's output enters in equation (3.8) with a negative sign. The intuition is that production takes place in the morning while ignoring future terms of trade and, therefore, total costs depend on beliefs. Simultaneously, low expectations on j's output allow island i to improve her terms of trade. Therefore, what islands expect the trading partner's output to be has a negative impact on their profits and here the incentive to lower each one's own beliefs arises.

In a certain period t, an island can meet all the three types of trading partners and with each of them it will earn different profits. The population of our economy is made of neutral, pessimistic and optimistic islands, each with its own share of population: n_n, n_p and n_o, respectively[18]. Thus, expected profits of an island i are given by the sum of the expected profits that it earns trading with each expectation type multiplied by its shares.

[16] Recalling from Proposition 1 that $\mathbb{E}_n(y_j^\eta) = \mathbb{E}(y_n^\eta)$

[17] See section A.2 for its derivation.

[18] Population shares vary over time; however, in the equations below, we ignore the subscript t to lighten notation.

Thus, considering a neutral i, her expected profits are given by

$$\begin{aligned}\mathbb{E}(\pi_{n,t}) = \mathbb{E}\left(y_n^{(1-\eta)}\right)\mathbb{E}(y_n^{\eta})\Big[& n_p\left[(1-\delta)^{\theta\eta\alpha}-1\right] \\ & + n_n\left[1-1\right] \\ & + n_o\left[(1+\delta)^{\theta\eta\alpha}-1\right]\Big]\end{aligned} \tag{3.9}$$

Similarly, we derive expected profits of pessimists and optimists[19].

In what follows we use the notation $\bar{\pi}_{r,s}$ to indicate the expected profits of an island of type 'r' when it trades with an island of type 's', with $r, s = \{p, n, o\}$.

Expected profits of neutral islands trading with neutral islands It follows from (3.8) and $\mathbb{E}_n(y_j^{\eta}) = \mathbb{E}(y_n^{\eta})$ that neutral islands earn zero expected profits when matched with another neutral agent, i.e.

$$\bar{\pi}_{n,n} = 0. \tag{3.10}$$

The reason is that, when an island correctly guesses the trading partner's output distribution, their interaction is similar to that among agents in a perfectly competitive economy and, thus, firms will produce until their marginal cost equals the expected price – which, under rational expectations, is known – which, in turn, equals firms' average costs, driving profits to zero. Under incomplete information, instead, there are opportunities for positive profits depending on islands' beliefs, as shown in equation (3.8).

3.2 Switching mechanism

We capture the sentiment dynamics in the economy through a switching model of beliefs similar to Lux (1995), in which we define the transition rates[20] from one attitude to another. We are interested in the dynamics of the shares of the different types of island, that is, $\dot{n}_p$, $\dot{n}_n$ and $\dot{n}_o$. Since we are working with an infinitely large population, the random effects disappear and we obtain a deterministic formulation of the dynamics of the inflows and outflows into and from the different attitudes. In particular:

$$\dot{n}_p = n_n q_{np} + n_o q_{op} - n_p q_{pn} - n_p q_{po}, \tag{3.11}$$

$$\dot{n}_n = n_p q_{pn} + n_o q_{on} - n_n q_{np} - n_n q_{no}, \tag{3.12}$$

[19] See section A.3 in Appendix A.

[20] For reasons of analytical tractability we use a continuous time formulation of the switching model, which is interpreted as the limit of the discrete time version with the length of a period Δt going to zero. See section A.4 of the Appendix for the derivation of our continuous dynamics model from its discrete time version.

$$\dot{n}_o = n_n q_{no} + n_p q_{po} - n_o q_{on} - n_o q_{op}, \tag{3.13}$$

where q_{np} and q_{pn} are the transition rates from the neutral belief to the pessimistic belief and *vice versa*, respectively; the other switching rates are defined analogously.

The idea is that the share of pessimistic islands in one period is given by the inflows from the two other types into the neutral group, minus the outflows from the neutral attitude to the two other types. In the same way, the share of neutral and optimistic islands are defined.

The transition rates on which the inflows and outflows depend have to reflect the fundamental factors driving the dynamics of the different attitudes in the economy: the economic payoffs of the different rules and the observation of what the others' beliefs are – as mentioned, both of them are public knowledge. For example, q_{pn}, the transition rate from the pessimistic to the neutral belief, must reflect the fact that the higher the difference of expected profits made by neutral and pessimistic islands, or the higher the difference between the shares of the neutral and pessimistic beliefs, the more likely it is that a pessimistic island turns neutral. Therefore, q_{pn} must be positively related to $\mathbb{E}(\pi_{n,t}) - \mathbb{E}(\pi_{p,t})$ and to $n_n - n_p$. These same variables should enter with a negative sign into q_{np}, the transition rate from the neutral to the pessimistic belief.

Regarding the functional relationship between the mentioned variables and the transition rates, we choose a formulation where the relative changes of the transition rates with respect to the variables mentioned above are linear and symmetrical, following Weidlich and Haag (1983) and Lux (1995). This means that, considering a general transition rate $q_{rs}(z)$ from a belief 'r' to a belief 's', which depends on a variable z, we want that $\frac{dq_{rs}(z)/dz}{q_{rs}(z)} = Az$ and $\frac{dq_{sr}(z)/dz}{q_{sr}(z)} = -Az$, for some constant A. Furthermore, the transition rates should include a parameter for the speed of switching, v, which guarantees that some changes in the beliefs happen even when the difference between expected profits of the two types and the difference between their shares equal zero: $q_{rs}(z) = q_{sr}(z) = v > 0$. Changes in the population shares driven by v can be interpreted as due to factors not taken into account by the model. In what follows we use $\bar{\pi}_{r-s,t}$ to identify the difference between the expected profits made by r and those made by s. Combining all the considerations mentioned above, we get the following transition rate:

$$q_{rs}(n_p, n_n, n_o) = v \exp(a_0 \bar{\pi}_{r-s,t}(n_p, n_n, n_o) + a_1(n_s - n_r)) \quad \text{for } r \to s, \tag{3.14}$$

where $r, s = \{p, n, o\}$.

The coefficient a_0 represents the strength of the impact of the difference between the two types' expected profits on the transition rates. The higher a_0, the more importance is given by the islands to the profits earned by each type. a_1 is the impact of the composition of the different beliefs in the economy on the transition rates. This parameter measures the importance of social influence in the model. The higher a_1, the more attention islands give to others' attitude and the more likely it is that each island imitates the predominant belief. When a_0 and a_1 are 0, the transition rates in both directions equal v, which means that they are determined only by the speed of switching.

4 Sentiment dynamics

4.1 No social influence

As a benchmark we first study the sentiment dynamics in the absence of social influence.

Since restricting the attention to two types increases the analytical tractability of the model, as a first step we consider an economy composed only by neutral and pessimistic islands.

4.1.1 Pessimistic and neutral expectations

In an economy with two types of belief, the composition of the population can be expressed by an 'opinion index', that is, the difference between the share of neutral and pessimistic islands: $x = n_n - n_p$. $x \in [-1, 1]$; therefore, if $x = 0$, the economy is in a balanced situation, $x > 0$ represents an economy characterized by a predominance of neutral islands and $x < 0$ implies that more islands are pessimistic. The extreme cases are $x = -1$, where all islands are pessimistic, and $x = 1$ where all islands are neutral and, importantly, all agents in the economy have rational expectations.

In order to analyze the dynamics of the population shares of such an economy, we can explore the dynamic behavior of x in a one-dimensional system. Considering that $\frac{x+1}{2} = n_n$ and $\frac{1-x}{2} = n_p$, we can write it as follows:

$$\dot{x} = (1 - x)q_{pn} - (1 + x)q_{np}. \tag{4.1}$$

The transition rates can be expressed as functions of x as well:

$$\begin{aligned} q_{pn}(x) &= v \exp(a_0 \bar{\pi}_{n-p,t}(x) + a_1 x) && \text{for pessimistic to neutral;} \\ q_{np}(x) &= v \exp(-a_0 \bar{\pi}_{n-p,t}(x) - a_1 x) && \text{for neutral to pessimistic.} \end{aligned} \tag{4.2}$$

If we plug (4.2) in (4.1), recalling that $\exp(y) - \exp(-y) = \sinh(y)$, $\exp(y) + \exp(-y) = \cosh(y)$ and $\frac{\sinh(y)}{\cosh(y)} = \tanh(y)$, we obtain the following differential equation:

$$\dot{x} = 2v[\tanh(a_0 \bar{\pi}_{n-p}(x) + a_1 x) - x] \cosh(a_0 \bar{\pi}_{n-p}(x) + a_1 x). \tag{4.3}$$

Absence of social influence implies that $a_1 = 0$ and the switching mechanism is driven only by the difference in expected profits made by neutral and pessimistic islands. Figure

C.1 in Appendix C shows this profit difference for different parameter constellations.

Difference between expected profits of neutral and pessimistic islands The difference between the expected profits of a neutral and a pessimistic island, $\bar{\pi}_{n-p}(x)$, is a linear function of x and it is given by

$$
\begin{aligned}
\bar{\pi}_{n-p,t}(x) = \mathbb{E}\left(y_n^{(1-\eta)}\right)\mathbb{E}(y_n^{\eta})\frac{1-x}{2}&\left[(1-\delta)^{\theta\eta\alpha}-1\right] \\
+&\frac{x+1}{2}\left[1-1\right] \\
-&\frac{1-x}{2}(1-\delta)^{\theta(1-\eta)\alpha}\left[(1-\delta)^{\theta\eta\alpha}-(1-\delta)\right] \\
-&\frac{x+1}{2}(1-\delta)^{\theta(1-\eta)\alpha}\left[1-(1-\delta)\right]\Bigg].
\end{aligned}
\tag{4.4}
$$

From this we obtain the following proposition:

Proposition 2 *For $\delta > 0$, expected profits of a pessimistic island are always higher than those of a neutral island.*

$$
\bar{\pi}_{n-p}(x) < 0, \qquad \text{for any } x. \tag{4.5}
$$

In fact, the profit function of equation (3.8) shows that, in order for the profits to be positive, the belief of i should be lower than j's expected output. Consider the case in which island i is pessimistic and trades with j which is neutral: we have that $\mathbb{E}_n(y_j^{\eta}) > \mathbb{E}_n(y_j^{\eta})(1-\delta)$. This means that pessimists on average are better off than neutral islands because they underestimate neutral islands' expected production. This allows them to simultaneously improve their terms of trade and reduce their costs by producing less. Here lies the basis of the trade off between individual and social outcomes, as we will deepen in section 5. Interestingly, the same happens also when a pessimist trades with another pessimist: $\mathbb{E}_n(y_j^{\eta})(1-\delta)^{\alpha\theta\eta} > \mathbb{E}_n(y_j^{\eta})(1-\delta)$, which is always true because $\eta\theta\alpha$ is smaller than one. In other words, the latter result implies that pessimistic islands are 'too pessimistic', since they systematically underestimate even the pessimists' production.

For $\delta \geq 0$, $\bar{\pi}_{n-p}(x)$ is an non decreasing function of x: although both types of island are better off – *ceteris paribus* – by meeting a neutral island rather than a pessimistic one, the advantage for neutral islands exceeds that for pessimistic ones. Thus, when all islands in the economy are neutral, that is, for $x = 1$, the absolute value of the difference of expected profits of neutral and pessimistic islands is minimized.

Figure 4.1 illustrates the sentiment dynamics of (4.3) in the absence of social influence, for different levels of a_0 and for the parameters shown in table 4.1, which are consistent with a more sophisticated version of the present model – without endogenous beliefs and

Table 4.1: Parameters of the model

Parameter	Value
K	1
Θ	0.65
ε	2
$\theta \equiv \frac{\Theta}{\varepsilon}$	0.325
η	0.5
δ	0.3
σ	0.038
υ	0.5

with capital accumulation –, presented by Angeletos and La'O (2013) as a variant of the Real Business Cycle model[21]. All the dynamics and variables presented in the main body of the paper are studied under this set of parameters. In Appendix C the robustness of results with respect to parameter variations is studied.

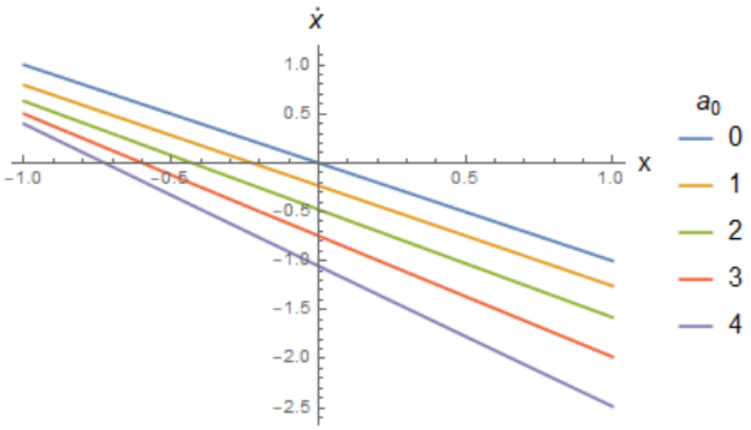

Figure 4.1: Dynamics of $\dot{x}$ without social influence. See the equilibrium points in table 4.2.

Steady state Without social influence, (4.3) is continuous and monotonically decreasing; in $x = -1$ it is positive and in $x = 1$ it is negative, therefore the solution exists, it is unique and it is globally stable. In particular, the fixed point is given by:

$$\tanh(a_0 \bar{\pi}_{n-p}(x)) = x, \tag{4.6}$$

which implies either $a_0 = 0$ and $x = 0$ or, if $a_0 \neq 0$, it must hold that $\frac{e^{a_0 \bar{\pi}_{n-p}(x)} - e^{-(a_0 \bar{\pi}_{n-p}(x))}}{e^{a_0 \bar{\pi}_{n-p}(x)} + e^{-(a_0 \bar{\pi}_{n-p}(x))}} = x$, which has no analytical solution. However, we can see from figure 4.1 that for $a_0 = 0$

[21] These parameters are also consistent with King and Rebelo (2000), except for δ, η and υ. In their model, $\eta = 1$, δ and υ are not included since they assume rational expectations and do not have differential equations. From these parameters it follows that: $K1 \equiv (\Theta^{\theta})(K^{(1-\Theta)}) = 0.869$, $\alpha \equiv \frac{1}{1-\theta+\eta\theta} = 1.194$, $\gamma \equiv \frac{1}{1-\theta} = 1.481$ and the degree of strategic complementarity $\eta\alpha\theta = 0.194$.

Table 4.2: Equilibrium points of the dynamics shown in figure 4.1.

	Equilibrium points	Derivative at the equilibrium
$a_0 = 0$	$x^* = 0$	$\frac{d(\dot{x})}{dx}\|_{x^*} =$ -1
$a_0 = 1$	$x^* =$ -0.2259	$\frac{d(\dot{x})}{dx}\|_{x^*} =$ -1.0248
$a_0 = 2$	$x^* =$ -0.4305	$\frac{d(\dot{x})}{dx}\|_{x^*} =$ -1.1046
$a_0 = 3$	$x^* =$ -0.5991	$\frac{d(\dot{x})}{dx}\|_{x^*} =$ -1.2445
$a_0 = 4$	$x^* =$ -0.7274	$\frac{d(\dot{x})}{dx}\|_{x^*} =$ -1.4523

the economy is in a balanced situation and the dynamics is driven by v alone, which is the speed of switching, whose idea is to incorporate those reasons for switching not included in the model. For $a_0 \geq 0$, the dynamics will converge to a pessimistic equilibrium, far from the rational expectations steady state. Therefore, we have the following proposition:

Proposition 3 *Rational expectations are not stable in our trade economy characterized by strategic complementarity and uncertainty.*

The reason for this shift away from $x = 1$ lies in the fact that, for $a_0 \neq 0$, islands take into account that the pessimistic belief implies higher expected profits than the neutral one. Furthermore, even for $a_0 = 0$, $x^* = 1$ is unstable because of the transition rates which are symmetric and linear in their relative changes with respect to each other. Therefore, in the absence of factors (unequally) influencing the switching between the different expectation types, the system converges to a balanced equilibrium where the shares of neutral and pessimistic islands are equal[22].

4.1.2 Pessimistic, neutral and optimistic expectations

In an economy which includes optimistic islands as well, the population composition can be represented by the shares of optimists and of pessimists: n_o and n_p, respectively. n_n, instead, is given by $1 - n_p - n_o$. The transition rates are of the form shown in equation (3.14) with $a_0 = 0$ and the dynamics of n_p and n_o is given by:

[22]The speed of switching v does not influence the converge of the system towards a balanced situation for $a_0 = 0$ because it equally affects all transition rates.

$$
\begin{aligned}
\dot{n}_p &= (n_n)v\exp(-a_0\bar{\pi}_{n-p,t}(n_p,n_o)) + n_o v\exp(-a_0\bar{\pi}_{o-p,t}(n_p,n_o)) \\
&\quad - n_p v\exp(a_0\bar{\pi}_{n-p,t}(n_p,n_o)) - n_p v\exp(a_0\bar{\pi}_{o-p,t}(n_p,n_o)); \\
\dot{n}_o &= (n_n)v\exp(a_0\bar{\pi}_{o-n,t}(n_p,n_o)) + n_p v\exp(a_0\bar{\pi}_{o-p,t}(n_p,n_o)) \\
&\quad - n_o v\exp(-a_0\bar{\pi}_{o-n,t}(n_p,n_o)) - n_o v\exp(-a_0\bar{\pi}_{o-p,t}(n_p,n_o)).
\end{aligned} \tag{4.7}
$$

Figure 4.2 shows the isoclines of such a system with different levels of a_0. The equilibrium points are listed in table 4.3.

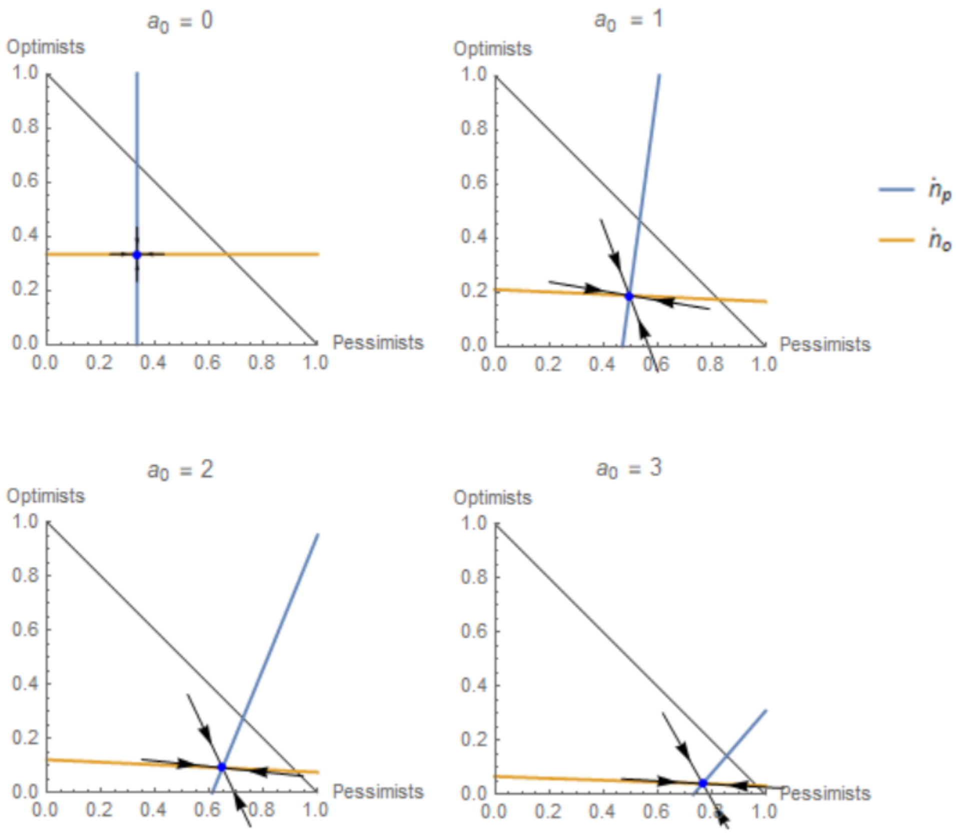

Figure 4.2: Phase portrait of the dynamics for the parameters listed in table 4.1, without social influence and for different a_0's. Inward arrows indicate the stable manifolds.

In the absence of social influence, the two-dimensional system behaves similarly to the one-dimensional dynamics. With no sensitivity to economic outcomes, the system converges to a situation in which the same number of islands are neutral, optimistic and pessimistic. As a_0 increases, more and more islands become pessimistic. The fixed point of the system is stable, as clear from the negative eigenvalues. The reason for the convergence to a situation dominated by pessimistic islands lies in expected profits of the different types. In particular, as shown in the two types setting, pessimistic islands

Table 4.3: Equilibrium points of the dynamics shown in figure 4.2.

	Equilibrium points	Eigenvalues	Eigenvectors
$a_0 = 0$	$(n_p^*, n_o^*) = (0.3333,\ 0.3333)$	λ_1 =-1.5 λ_2 =-1.5	$v_1 = (0,\ 1)$ $v_2 = (1,\ 0)$
$a_0 = 1$	$(n_p^*, n_o^*) = (0.4974,\ 0.1879)$	λ_1 =-1.8139 λ_2 =-1.3610	$v_1 =$ (-0.3524, 0.9359) $v_2 =$ (0.9860, -0.1666)
		λ_2 =-1.3438	$v_2 =$ (0.9948, -0.1015)
$a_0 = 3$	$(n_p^*, n_o^*) = (0.7667,\ 0.0408)$	λ_1 =-3.4151 λ_2 =-1.4233	$v_1 =$ (-0.4925, 0.8703) $v_2 =$ (0.9983, -0.0577)

earn higher expected profits than neutral ones, which holds true in the three expectations framework as well. Analogously, expected profits of neutral islands are greater than those of optimists, as clarified in what follows.

Difference between expected profits of optimistic and neutral agents The difference between expected profits of an optimistic and a neutral island, $\bar{\pi}_{o-n}(p, o)$, is given by

$$\begin{aligned}\bar{\pi}_{o-n,t}(n_p, n_o) = \mathbb{E}\left(y_n^{(1-\eta)}\right)\mathbb{E}(y_n^{\eta})\Big[& n_n(1+\delta)^{\theta(1-\eta)\alpha}\left[1-(1+\delta)\right] \\ & +n_o\ (1+\delta)^{\theta(1-\eta)\alpha}\left[(1+\delta)^{\theta\eta\alpha}-(1+\delta)\right] \\ & -n_n\left[1-1\right] \\ & -n_o\left[(1+\delta)^{\theta\eta\alpha}-1\right]\Big].\end{aligned} \tag{4.8}$$

It is worth noting the following proposition:

Proposition 4 *For $\delta > 0$, expected profits of a neutral island are always greater than those of an optimistic island.*

$$\bar{\pi}_{o-n}(n_p, n_o) < 0, \qquad \text{for any } n_p \text{ and } n_o. \tag{4.9}$$

In fact, first line of equation (4.8) shows that expected profits of an optimistic island trading with a neutral one are negative. Moreover, the expected profits of an optimist trading with another optimist are negative, too, because of the degree of strategic complementarity. In general, optimists overestimate expected output of every type of island and, thus, earn the lowest expected profits. It follows that expected profits of pessimists are also higher than those of optimists.

4.2 Dynamics with social influence

Let us consider the case in which islands observe what others think and do and eventually imitate them. In formal terms, this implies $a_1 > 0$, that is, social influence is positive and affects the transition rates.

4.2.1 Pessimistic and neutral expectations

The dynamics of an economy populated by pessimistic and neutral islands is given by

$$\dot{x} = 2v[\tanh(a_0\bar{\pi}_{n-p}(x) + a_1 x) - x]\cosh(a_0\bar{\pi}_{n-p}(x) + a_1 x), \tag{4.10}$$

which is shown in figure 4.3, for $a_0 = 1$. The equilibrium points of the different dynamics shown here are listed in table 4.4

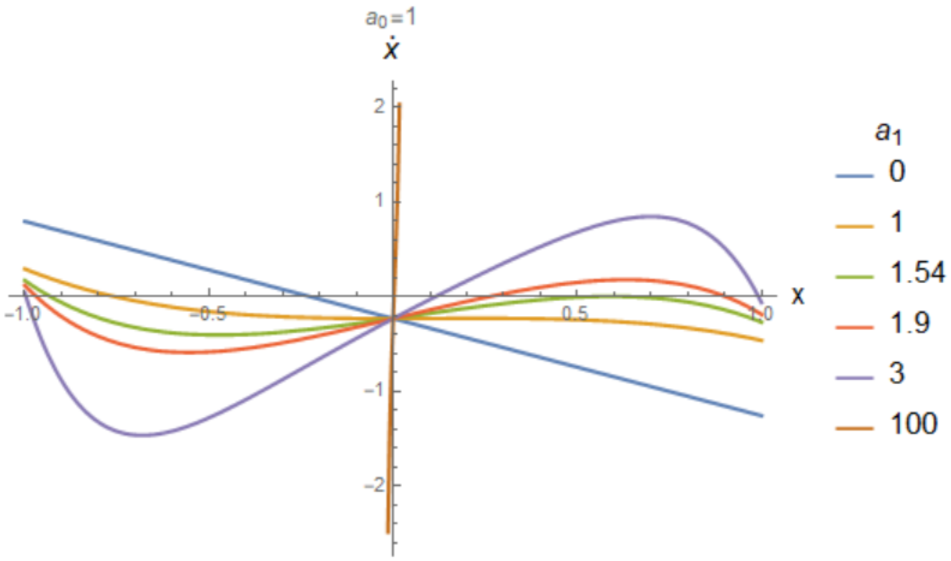

Figure 4.3: Dynamics of the economy with different levels of social influence; with the parameters shown in table 4.1.

Steady states The dynamics shown in equation (4.10) is a continuous function of x, which takes positive values for $x = -1$ and negative values for $x = 1$; therefore, it has at least one solution. It is not analytically solvable whether and for which values the dynamics is monotonically decreasing.

Table 4.4: Equilibrium points of the dynamics shown in figure 4.3.

	Equilibrium points	Derivative at the equilibrium
$a_1 = 0$	x^* = -0.2259	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -1.0248
$a_1 = 1$	x^* = -0.7560	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -0.8720
$a_1 = 1.5357$	x_1^* = -0.9302 $x_2^* = 0.5893$	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -2.1593 $\frac{d(\dot{x})}{dx}\|_{x_2^*} = 0.0000$
$a_1 = 1.9$	x_1^* = -0.9688 $x_2^* = 0.2613$ $x_3^* = 0.9029$	$\frac{d(\dot{x})}{dx}\|_{x_1^*}$ = -3.5613 $\frac{d(\dot{x})}{dx}\|_{x_2^*} = 0.7998$ $\frac{d(\dot{x})}{dx}\|_{x_3^*}$ = -1.5088
$a_1 = 3$	x_1^* = -0.9968 $x_2^* = 0.1149$ $x_3^* = 0.9918$	$\frac{d(\dot{x})}{dx}\|_{x_1^*}$ = -12.3168 $\frac{d(\dot{x})}{dx}\|_{x_2^*} = 1.9753$ $\frac{d(\dot{x})}{dx}\|_{x_3^*}$ = -7.4532
$a_1 = 100$	x_1^* = -1 $x_2^* = 0.0053$ $x_3^* = 1$	$\frac{d(\dot{x})}{dx}\|_{x_1^*} = -2.3856 * 10^43$ $\frac{d(\dot{x})}{dx}\|_{x_2^*} = 99.0514$ $\frac{d(\dot{x})}{dx}\|_{x_3^*} = -8.4157 * 10^42$

We can study its behavior by analyzing its representation in figure 4.3, where $a_0 = 1$ and a_1 varies as shown. The blue line represents the case in which there is no social influence; here the solution is unique, x^* = -0.23, the fixed point is globally stable and the economy is made of 38.71% neutral islands and 61.30% pessimistic islands. By increasing the importance of social influence at $a_1 = 1$, the unique stable equilibrium shifts to the left, leading even more islands to become pessimistic. In this case, x^* = -0.756, which implies that 87.8% of islands are neutral and the rest is pessimistic; social influence, in this case, amplifies the number of pessimistic islands.

However, if we increase social influence to 1.5357, as the green line of figure 4.3 shows, the negative stable equilibrium shifts even more to the left and we observe a qualitative change in the dynamics. In fact, a new equilibrium arises, $x_2^* = 0.59$, stable from the right hand side and unstable from the left hand side, in which $\dot{x}$ is flat. This is a critical point in which a fold bifurcation occurs; we observe that with $a_1 = 1.9$, the system has three equilibria. The negative one is x_1^* =-0.97, locally stable, where almost all islands are pessimistic; on the positive x-axis, there are two equilibria. One is $x_2^* = 0.2613$, characterized by 63.07% neutral islands; it is unstable. $x_3^* = 0.9029$ implies 95.15% neutral islands, that is, it gets closer to the rational expectations scenario and it is locally stable. This type of dynamics states that, assuming that the economy is initially made of half neutral and half pessimistic islands, with time more neutral islands will turn pessimistic until the latter type constitutes 98.44% of the population. However, if the initial situation is characterized by, e.g., 70% neutral islands, over time the economy will be composed almost only of them.

With further increments in a_1, the dynamics does not change qualitatively anymore; in fact, with $a_1 = 3$, there are again three fixed points of the same nature as in the previous case. With a growing social influence, what changes is that two locally stable fixed points converge towards the uniform states $x^* = -1$ and $x^* = 1$ respectively, and the unstable fixed point, which is the boundary of the basins of attraction of the stable ones, moves towards $x^* = 0$, as shown by part of the dynamics under $a_1 = 100$ in figure 4.3.

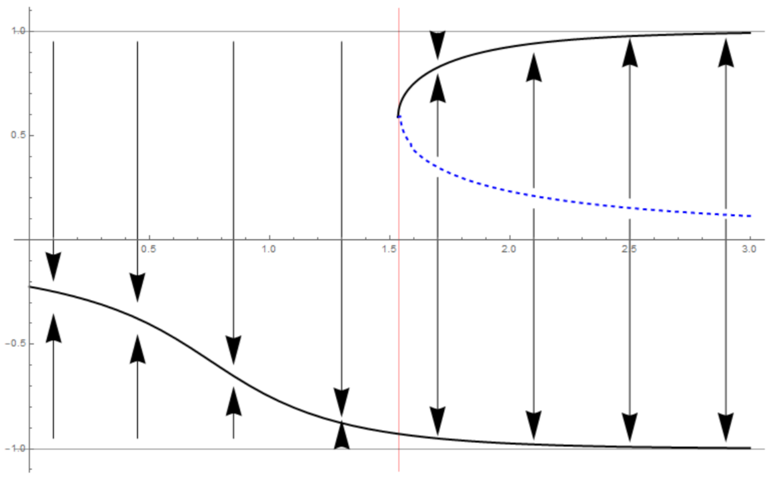

Figure 4.4: Bifurcation Diagram of the system with the parameters shown in table 4.1 and $a_0 = 1$.

The fold bifurcation can be observed more clearly in the bifurcation diagram in figure 4.4, where the black lines show how stable steady states evolve with the social influence parameter. The blue dotted line illustrates the development of the unstable fixed point. The red vertical line shows the critical value of a_1, around 1.5357, where two new equilibria arise. This figure represents clearly the possible convergence of agents to the rational expectations equilibrium, defined by $x = 1$.

4.2.2 Pessimistic, neutral and optimistic expectations

In an economy with three types of belief and a positive social influence, the dynamics of the population shares are of the form of equation (3.14) with $a_1 > 0$. The phase portrait of this system, for $a_0 = 1$ and $a_1 = 1$, is shown in figure 4.5. There is a unique stable equilibrium; here, the economy is composed of 79.66% pessimistic islands, 7% optimistic islands and the rest is neutral. Therefore, social influence shifts the equilibrium towards an economy with more pessimists, as occurs in an economy composed of two types only.

The dynamics with higher levels of social influence is shown in figure 4.6, the equilibrium points are listed in table C.7 in Appendix C.

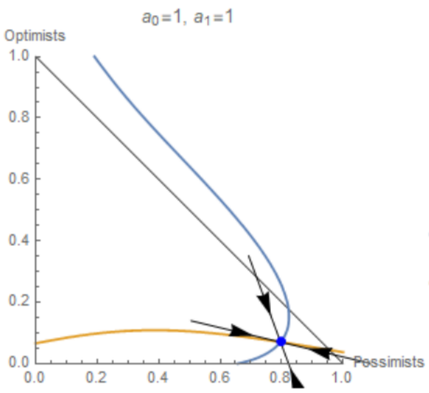

Figure 4.5: Dynamics of the two-dimensional system for the parameters listed in table 4.1, for $a_0 = 1$ and $a_1 = 1$. Inward arrows indicate stable manifolds.

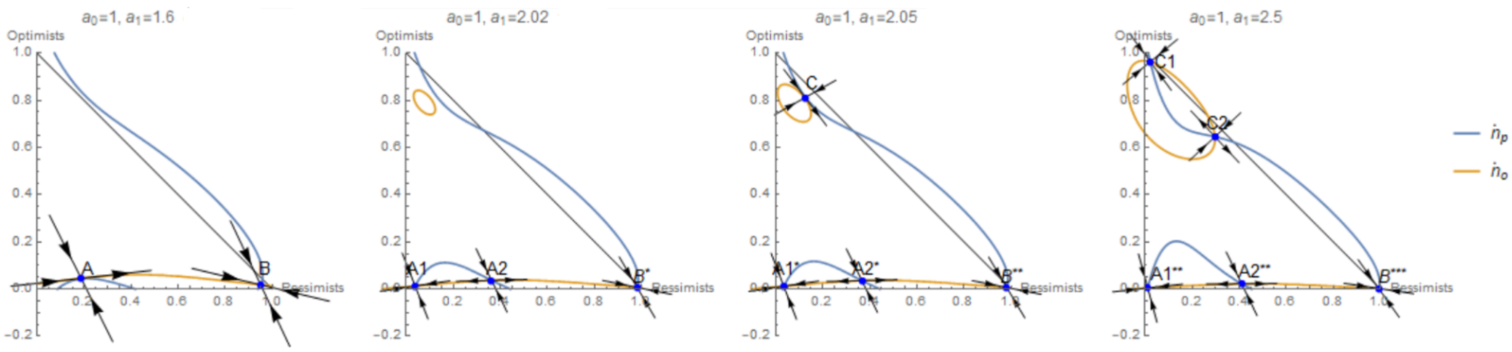

Figure 4.6: Dynamics of the two-dimensional system for the parameters listed in table 4.1, for different levels of social influence and $a_0 = 1$. Inward and outward arrows indicate stable and unstable manifolds, respectively.

With a social influence parameter of 1.6, while the equilibrium point characterized by a predominance of the pessimistic belief shifts towards the right, a new fixed point emerges, $A = (0.1873, 0.0455)$, characterized by a majority of neutral islands. This effect corresponds to that emerging in the one-dimensional system. The value $a_1 = 1.6$ is a bifurcation point: as the importance of social influence grows, a fold bifurcation occurs and A splits in two fixed points. In particular, for $a_1 = 2.02$, the two new equilibria are $A1 = (0.0376, 0.0129)$ and $A2 = (0.3624, 0.0367)$. The former is very close to the case in which both optimistic and pessimistic islands are almost zero, which implies a predominance of neutral islands; it is a stable fixed point. The latter, instead, is a saddle node which separates the basins of attraction of the other two stable equilibrium points.

Expanding social influence to $a_1 = 2.05$, the dynamics qualitatively changes again and we observe another fold bifurcation. In particular, a fourth equilibrium point emerges at $C = (0.1256, 0.8118)$, which is the first equilibrium point characterized by a predominance of optimistic islands. The other two stable points keep shifting towards the extreme scenarios of an economy in which all islands are either pessimistic or optimistic; this is the effect that any increment in the level of social influence has on them. At the saddle point $A2^*$, there are more pessimistic islands than in $A2$, which implies that the basin of attraction to $A1^*$ is larger: a higher number of initial scenarios leads to the almost-rational expectations case. In figure 4.6, the last graph on the right show the dynamics for $a_1 = 2.5$, where point C has split in $C1$ and $C2$ after the fold bifurcation. $C1$ is stable and characterized by 96.33% optimistic islands. $C2$ is a saddle node which separates the basin of attraction of $C1$ from those of $A1^{**}$ and B^{***}.

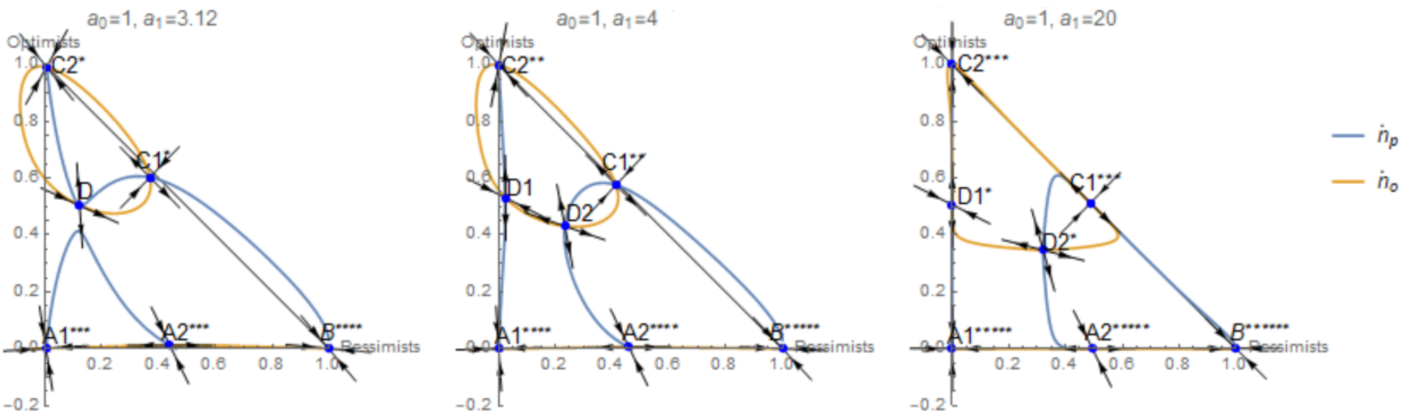

Figure 4.7: Dynamics of the two-dimensional system for the parameters shown in table 4.1, for different levels of social influence and $a_0 = 1$. Inward and outward arrows indicate stable and unstable manifolds, respectively.

Figure 4.7 shows the dynamics for higher levels of social influence; table C.8 in Appendix C illustrates its fixed points. In particular, $a_1 = 3.12$ is another bifurcation point, in which the new fixed point D arises and then, for $a_1 = 4$, splits in $D1$, saddle node, and $D2$, unstable node. $D1 = (0.0241, 0.5309)$ is a saddle node, which, together with points $D2$ and $C1^*$, delimits the area converging to the optimistic equilibrium. Moreover, the quasi-vertical eigenvector of point $D2$ separates the basins of attraction to $A1^{****}$ and B^{*****}. Now the fixed points are seven and the dynamics does not change anymore qualitatively for higher levels of social influence. In fact, the graph on the right of figure 4.7, for $a_1 = 20$, shows again seven equilibrium points. There are three stable nodes, which are characterized by all islands being neutral, pessimistic or optimistic. The other four fixed points divides the simplex in the different basins of attraction; the higher a_1, the more these basins of attraction become equal to each other. In Appendix C we show how the dynamics vary under different parameters. In particular, higher levels of a_0 require higher levels of social influence for the various bifurcations to occur. The idea is that the more attention islands give to the difference of expected profits between the groups, the higher is the tendency to become pessimistic and the stronger social influence must be in order for the new equilibria to arise.

5 Social influence as a coordination device and its impact on production and welfare

Social influence is crucial in driving the dynamics of our system and the efficiency and stability of the economy. In fact, in the absence of social influence, when islands are only sensitive to profits, both with and without optimistic islands, agents deviate from rational expectations and coordinate on a pessimistic equilibrium. As the importance of social influence grows, that pessimistic steady state gets even more pessimistic; however, as the social influence parameter crosses a certain threshold, a more neutral fixed point emerges. A further increase, in the two-dimensional dynamics, leads to the emergence of an optimistic equilibrium. The higher social influence, the more similar in size the basins of attraction to the different stable points become. Hence, social influence appears as a counteracting force that balances individual incentives to lower expectations and output. Importantly, certain levels of social influence make it possible for agents to converge to where the hierarchy of higher-order beliefs collapses to rational expectations, i.e. to the true fundamental.

But what about social outcomes such as production and welfare: how are they affected by social influence?

Production Y_t is the production of the whole economy in t. Expected production is given by the sum of each type's expected output:

$$\mathbb{E}(Y_t) = n_p\,\mathbb{E}(y_n)(1-\delta)^{\theta\alpha} + n_n\,\mathbb{E}(y_n) + n_o\,\mathbb{E}(y_n)(1+\delta)^{\theta\alpha}. \tag{5.1}$$

It is maximized when all islands are optimistic. Also, with a fixed amount of optimists, as neutral islands increase, output grows as well. Hence, the strength of social influence has a positive impact on production as long as it improves the chances for the population to coordinate on less pessimistic sentiments.

Welfare Welfare is measured as the expected utility of the whole economy. The utility of island i trading with island j is illustrated in equation (2.2), in which we substitute the equilibrium values of equation (2.4)[23]:

[23]See section A.5 for its derivation.

$$
\begin{aligned}
\mathbb{E}(U) = \mathbb{E}(y_n^{(1-\eta)})\,\mathbb{E}(y_n^{\eta})\Big[&(n_n)^2\,[1-\theta] \\
&+ (n_n n_p)\,\big[(1-\delta)^{\theta\eta\alpha}-\theta\big] \\
&+ (n_n n_o)\,\big[(1+\delta)^{\theta\eta\alpha}-\theta\big] \\
&+ (n_p n_n)(1-\delta)^{\theta(1-\eta)\alpha}\,[1-(1-\delta)\theta] \\
&+ (n_p)^2(1-\delta)^{\theta(1-\eta)\alpha}\,\big[(1-\delta)^{\theta\eta\alpha}-(1-\delta)\theta\big] \\
&+ (n_p n_o)(1-\delta)^{\theta(1-\eta)\alpha}\,\big[(1+\delta)^{\theta\eta\alpha}-(1-\delta)\theta\big] \\
&+ (n_o n_n)(1+\delta)^{\theta(1-\eta)\alpha}\,[1-(1+\delta)\theta] \\
&+ (n_o n_p)(1+\delta)^{\theta(1-\eta)\alpha}\,\big[(1-\delta)^{\theta\eta\alpha}-(1+\delta)\theta\big] \\
&+ (n_o)^2(1+\delta)^{\theta(1-\eta)\alpha}\,\big[(1+\delta)^{\theta\eta\alpha}-(1+\delta)\theta\big]\Big].
\end{aligned}
\tag{5.2}
$$

Equation (5.2) helps understanding the different composition effects on welfare. First line indicates the utility generated by a neutral island meeting another neutral island, times the product of their shares; second line displays the utility generated by a neutral island trading with a pessimistic island times the product of their shares and so on. In particular, equation (5.2) reveals that the following matches have an unambiguous positive effect on welfare:

- Neutral-Neutral;
- Neutral-Optimistic;
- Pessimistic-Neutral;
- Pessimistic-Pessimistic;
- Pessimistic-Optimistic.

However, for the following matches the effect is ambiguous:

- Neutral-Pessimistic;
- Optimistic-Neutral;
- Optimistic-Pessimistic;
- Optimistic-Optimistic.

The ambiguity, in all cases, depends mostly on the degree of optimism or pessimism, δ, and the amount of labor employed in the production, θ. For instance, concerning the Neutral-Pessimistic match of second line of equation (5.2): the expression in the square brackets is positive if $\delta < 1-\theta^{\frac{1}{\theta\eta\alpha}}$, and negative if the opposite is true. The higher θ and η, the smaller the room for δ to satisfy that inequality. We recall that $\theta \equiv \frac{\Theta}{\varepsilon}$, where Θ is the labor share of income and ε is the Frisch elasticity of labor supply: they are, respectively, positively and negatively related with the optimal amount of labor employed in the production. Thus, when neutral islands trading with pessimists obtain a negative expected utility, the reason is that they are working too much with respect to the expected production of the pessimists, which negatively depends on δ. A similar reasoning explains the ambiguity of the other matches; moreover, the more an agent produces, the greater is the impact of such an ambiguity on overall welfare. In fact, if θ is too high, the loss of overall welfare is generated proportionally to islands' production levels: an optimistic island will be the most responsible for it followed by neutral and pessimistic agents.

The considerations above explain why welfare is not necessarily monotone in the population shares, as illustrated in figure 5.1. The graph on the left displays an economy whose welfare linearly increases in the number of optimistic islands but even more in the number of neutral ones. An economy entirely populated of pessimists is the most inefficient scenario while rational expectations bring about the highest welfare. By increasing θ, welfare becomes slightly non linear in the population shares, but the relative efficiency of the three extreme points are unchanged. Finally, in the graph on the right welfare appears highly non monotonous and the more mixed composition of beliefs are the least efficient. Here, high θ translates in too much work and a loss of utility for those matches generating an ambiguous utility.

We find that the rational expectations scenario, is always the most efficient:

Result *Welfare is highest under rational expectations, that is, when all islands have neutral beliefs.*

For $\delta = 0$, the economy's welfare is given by the first line of equation (5.2) repeated nine times. For $\delta > 0$ we are back in the economy made of neutral, pessimistic and optimistic islands; given that the derivative of equation (5.2) with respect to δ is negative[24], it results that welfare is highest under rational expectations. Proposition 5 suggests that coordination issues generate inefficiencies and cause welfare losses.

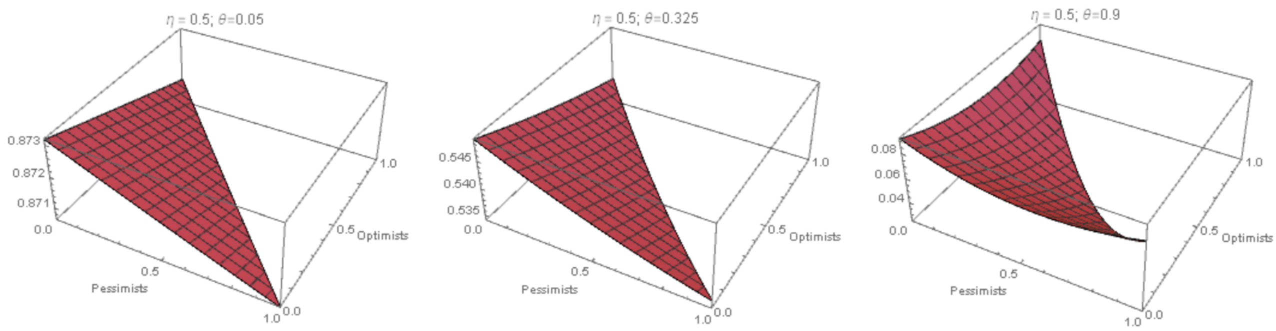

Figure 5.1: Welfare for an economy populated by pessimistic, neutral and optimistic islands.

Therefore, as regards what impact the importance of social influence has on production and welfare, it depends on how social influence affects the population shares. Table 5.1 shows the amount of production and welfare at each equilibrium point for the one-dimensional economy represented in figure 4.3. Columns Output and Welfare present the

[24] This result is proven numerically for our parameters.

expected values of such figures for the entire population, that we compute by multiplying values in Eq. Output and Eq. Welfare by the probabilities for the economy to reach such stable steady states, assuming that any initial state is equally likely[25]. In particular, we notice that as the importance of social influence grows from 0 to 1.9, expected output declines from 0.7492 to 0.7481; as a_1 exceeds 1.9, its effect on output is positive, raising production to 0.7549 and 0.7608. In terms of welfare, the impact is strongly positive as the parameter a_1 is raised.

Table 5.2 illustrates the impact of social influence on the economy populated by optimistic islands as well, whose dynamics is shown in 4.5 and 4.1. We do not measure the areas converging to the different steady states and, therefore, expected output and welfare. Nevertheless, the figures there point out to a negative effect of social influence on output at the pessimistic steady state, which declines from 0.7328 to 0.7085. However, it is counterbalanced and probably reversed as in the one dimensional case by the output growth at the neutral- (from 0.7959 to 0.8136) and optimistic-dominated (from 0.8349 to 0.8561) equilibria. In terms of welfare, social influence leads to a worsening of the utility level characterizing the pessimistic fixed point, which declines by about 0.19%. At the same time, a stronger social influence drives agents to coordinate on more efficient scenarios taking place at the optimistic equilibria, characterized at every point by larger welfare levels than at the pessimistic fixed point. Also, it generates the more neutral steady state, at which welfare is highest. As social influence grows, the basins of attraction of the three different steady states get more and more similar in size. Thus, for $a_1 = 20$, about one third of initial states lead to each equilibrium point, so that expected welfare is around 0.5411, 0.9% higher than the economy's welfare in the absence of social influence. To sum up, with the set of parameters indicated in table 4.1 we find that the effect of social influence on expected utility overall is positive, with welfare being highest under rational expectations.

6 Conclusions

In the present paper, we incorporate endogenously emerging beliefs driven by economic outcomes and social influence, in a simple Islands model, characterized by uncertainty and strategic complementarity among agents which generate a hierarchy of beliefs and coordination issues. We aim at studying the impact of social influence on the stability and efficiency of the economy.

We find that social influence is a powerful mechanism which modifies stability and efficiency of such an economy. In its absence, incomplete knowledge and individual incentives drive agents to converge on a pessimistic inefficient equilibrium and let frictional coordination cause welfare losses. This occurs because individuals, in a context of un-

[25]See Appendix A for details.

Table 5.1: Production and welfare of the one-dimensional system with the parameters shown in table 4.1, with $a_0 = 1$ and different levels of social influence.

	Equilibrium points	Eq. output	Exp. Output	Eq. welfare	Exp. Welfare
$a_1 = 0$	$x^* =$ -0.2259	0.7492	0.7492	0.5278	0.5278
$a_1 = 1$	$x^* =$ -0.7560	0.7213	0.7213	0.5313	0.5313
$a_1 = 1.5357$	$x_1^* =$ -0.9302 $x_2^* =$ 0.5893	0.7121 0.7920	0.7121	0.5341 0.5373	0.5341
$a_1 = 1.9$	$x_1^* =$ -0.9688 $x_2^* =$ 0.2613 $x_3^* =$ 0.9029	0.7101 0.7748 0.8085	0.7481	0.5348 0.5313 0.5457	0.5385
$a_1 = 3$	$x_1^* =$ -0.9968 $x_2^* =$ 0.1149 $x_3^* =$ 0.9918	0.7086 0.7671 0.8132	0.7549	0.5354 0.5295 0.5486	0.5412
$a_1 = 100$	$x_1^* =$ -1 $x_2^* =$ 0.0053 $x_3^* =$ 1	0.7085 0.7613 0.8136	0.7608	0.5355 0.5286 0.5489	0.5422

certainty and because of strategic complementarity, are unable to coordinate well their beliefs and actions, and thus are driven to lower their beliefs in order to improve their terms of trade and reduce their costs. However, this negatively affects social outcomes. With social influence, instead, agents, by imitating each other, become able to coordinate on other and eventually more efficient stable steady states. The impact of social influence on such a tendency might be interpreted through two lenses: on the one hand, it abates the role of uncertainty by encouraging agents to interact and imitate each other. In fact, even though individuals are not aware of their exact match, they try to learn the predominant opinion from the majority and act accordingly. Interestingly, as social influence crosses a certain threshold, the neutral equilibrium emerges where agents are able to coordinate on the rational expectations steady state: here, the economy behaves like under complete knowledge and the hierarchy of beliefs collapses to the true fundamental. On the other hand, social influence smooths the trade off between individual incentives and social outcomes, by nudging individuals to put less weight on their own payoffs in order to conform with the majority. With a closer look, this dual interpretation may remind the double motivation identified by psychologists for the role of social influence mentioned in the Introduction: one based on the goal to form a correct interpretation of reality and the other grounded on the desire of social approval. We conclude by pointing out the role of social influence as a coordination device in a macroeconomic context.

Table 5.2: Production and welfare for different levels of social influence.

	Equilibrium points	Production	Welfare
$a_1 = 1$	$(n_p^*, n_o^*) = (0.7966,\ 0.070)$	0.7328	0.5365
$a_1 = 1.6$	$A = (0.1873,\ 0.0455)$	0.7959	0.5453
	$B = (0.9503,\ 0.0180)$	0.7144	0.5357
$a_1 = 2.02$	$A1 = (0.0376,\ 0.0129)$	0.8102	0.5482
	$B^* = (0.9804,\ 0.0072)$	0.7108	0.5356
$a_1 = 2.05$	$A1^* = (0.0348,\ 0.0120)$	0.8105	0.5482
	$B^{**} = (0.9817,\ 0.0068)$	0.7107	0.5356
	$C = (0.1256,\ 0.8118)$	0.8349	0.5378
$a_1 = 2.5$	$A1^{**} = (0.0120,\ 0.0044)$	0.8125	0.5487
	$B^{***} = (0.9929,\ 0.0026)$	0.7093	0.5355
	$C1 = (0.0228,\ 0.9633)$	0.8522	0.5386
$a_1 = 3.12$	$A1^{***} = (0.0032,\ 0.0012)$	0.8133	0.5489
	$B^{****} = (0.9981,\ 0.0007)$	0.7087	0.5355
	$C2^* = (0.0055,\ 0.9910)$	0.8552	0.5389
$a_1 = 4$	$A1^{****} = (0.0005,\ 0.0002)$	0.8136	0.5490
	$B^{*****} = (0.9997,\ 0.0001)$	0.7085	0.5355
	$C2^{**} = (0.0009,\ 0.9985)$	0.8559	0.5389
$a_1 = 20$	$A1^{*****} = (0.,\ 0.)$	0.8136	0.5490
	$B^{******} = (1.,\ 0.)$	0.7085	0.5355
	$C2^{***} = (0.,\ 1.)$	0.8561	0.5389

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Appendices

Appendix A Derivation of formulas

A.1 Expected productivity

Islands' productivity is lognormally distributed: $A_i \sim \log N(0, \sigma_A)$, where $\sigma_A > 0$.

$$\mathbb{E}(A_i^{\eta\alpha}) = \int_0^{+\infty} A_i^{\eta\alpha} dF(A_i), \tag{A.1}$$

which, given that $dF(A_i) = f(A_i)dA_i$, becomes:

$$\mathbb{E}(A_i^{\eta\alpha}) = \int_0^{+\infty} A_i^{\eta\alpha} f(A_i)dA_i. \tag{A.2}$$

$f(A_i)$ is lognormal, i.e. $f(A_i) = \frac{1}{A_i\sqrt{2\pi\sigma^2}} e^{-\frac{\log A_i}{2\sigma^2}}$. Changing the variable $\tilde{A}_i = \log A_i$, we obtain:

$$\mathbb{E}(A_i^{\eta\alpha}) = \int_{-\infty}^{+\infty} e^{\eta\alpha\tilde{A}_i} \varphi(\tilde{A}_i)d\tilde{A}_i, \tag{A.3}$$

where $\varphi(\tilde{A}_i)$ is the Normal distribution of $\tilde{A}_i$. It follows that:

$$\mathbb{E}(A_i^{\eta\alpha}) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{\eta\alpha\tilde{A}_i - \frac{\tilde{A}_i}{2\sigma^2}} d\tilde{A}_i. \tag{A.4}$$

$e^{\eta\alpha\tilde{A}_i - \frac{\tilde{A}_i}{2\sigma^2}}$ can be transformed in a squared binomial by adding $e^{\frac{\sigma^2\eta^2\alpha^2}{2}}$:

$$\mathbb{E}(A_i^{\eta\alpha}) = e^{\frac{\sigma^2\eta^2\alpha^2}{2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\frac{\tilde{A}_i}{\sqrt{2\sigma^2}} - \frac{\sigma\eta\alpha}{\sqrt{2}})^2} d\tilde{A}_i. \tag{A.5}$$

Changing again a variable, i.e. $\frac{\tilde{A}_i}{\sqrt{2\sigma^2}} - \frac{\sigma\eta\alpha}{\sqrt{2}} = \frac{z}{\sqrt{2}}$, it follows that

$$\mathbb{E}(A_i^{\eta\alpha}) = e^{\frac{\sigma^2\eta^2\alpha^2}{2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \Phi(+\infty) = e^{\frac{\sigma^2\eta^2\alpha^2}{2}}. \tag{A.6}$$

A.2 Profits

Profits are given by the difference of revenues and costs: $\pi_i = p_i y_i - w_i l_i - r_i k_i$. Costs are obtained by considering the optimal quantities and prices of labor and land. As for labor, we have that wage must be equal to the marginal disutility of working and to the expected marginal revenues of labor:
$\frac{\delta y_{it}}{\delta l_{it}} = \frac{\delta(A_i(l_{it})^{\Theta}(k_{it})^{1-\Theta})}{\delta l_{it}} = \mathbb{E}_{it}[p_{it}]\Theta\frac{y_{it}}{l_{it}}$, therefore we have that $l^{\varepsilon-1} = \mathbb{E}_{it}[p_{it}]\Theta\frac{y_{it}}{l_{it}}$. From

the latter we can find the optimal level of labor $l_{it} = (\mathbb{E}_{it}[p_{it}]\Theta y_{it})^{\frac{1}{\varepsilon}}$. Since $w_{it} = l^{\varepsilon} - 1$, therefore the optimal $w_{it}l_{it} = l^{\varepsilon} = \mathbb{E}_{it}[p_{it}]\Theta y_{it}$.

As for land, the optimal r_{it} must be equal to the marginal utility of k_{it} which is given by $\frac{\delta y_{it}}{\delta k_{it}} = \frac{\delta(A_i(l_{it})^{\Theta}(k_{it})^{1-\Theta})}{\delta k_{it}} = \mathbb{E}_{it}[p_{it}](1-\Theta)\frac{y_{it}}{k_{it}}$, k_{it} in equilibrium is equal to K which is the fixed endowment of land and it is normalized to one, therefore the optimal $r_{it}k_{it} = \mathbb{E}_{it}[p_{it}](1-\Theta)y_{it}$.

We already know that $p_{it} = y_{it}^{-\eta}y_{jt}^{\eta}$, therefore $\mathbb{E}_{it}[p_{it}] = y_{it}^{-\eta}\,\mathbb{E}_{it}[y_{jt}^{\eta}]$. Therefore, profits are given by $p_{it}y_{it} - w_{it}l_{it} - k_{it}r_{it} = y_{it}^{1-\eta}y_{jt}^{\eta} - \mathbb{E}_{it}[p_{it}]\Theta y_{it} - \mathbb{E}_{it}[p_{it}](1-\Theta)y_{it} = y_{it}^{1-\eta}y_{jt}^{\eta} - \mathbb{E}_{it}[y_{jt}^{\eta}]\Theta y_{it}^{1-\eta} - \mathbb{E}_{it}[y_{jt}^{\eta}](1-\Theta)y_{it}^{1-\eta} = y_{it}^{1-\eta}y_{jt}^{\eta} - y_{it}^{1-\eta}\,\mathbb{E}_{it}[y_{jt}^{\eta}]$, or:

$$y_{it}^{1-\eta}(y_{jt}^{\eta} - \mathbb{E}_{it}(y_{jt}^{\eta})). \tag{A.7}$$

A.3 Expected profits of pessimistic and optimistic islands

For pessimists:

$$\begin{aligned}\mathbb{E}(\pi_{p,t}) = {} & n_p\,\mathbb{E}\left(y_n^{(1-\eta)}\right)(1-\delta)^{\theta(1-\eta)\alpha}\left[\mathbb{E}(y_n^{\eta})(1-\delta)^{\theta\eta\alpha} - \mathbb{E}(y_n^{\eta})(1-\delta)\right] \\ & + n_n\,\mathbb{E}\left(y_n^{(1-\eta)}\right)(1-\delta)^{\theta(1-\eta)\alpha}\left[\mathbb{E}(y_n^{\eta}) - \mathbb{E}(y_n^{\eta})(1-\delta)\right] \\ & + n_o\,\mathbb{E}\left(y_n^{(1-\eta)}\right)(1-\delta)^{\theta(1-\eta)\alpha}\left[\mathbb{E}(y_n^{\eta})(1+\delta)^{\theta\eta\alpha} - \mathbb{E}(y_n^{\eta})(1-\delta)\right].\end{aligned} \tag{A.8}$$

For optimists:

$$\begin{aligned}\mathbb{E}(\pi_{o,t}) = {} & n_p\,\mathbb{E}\left(y_n^{(1-\eta)}\right)(1+\delta)^{\theta(1-\eta)\alpha}\left[\mathbb{E}(y_n^{\eta})(1-\delta)^{\theta\eta\alpha} - \mathbb{E}(y_n^{\eta})(1+\delta)\right] \\ & + n_n\,\mathbb{E}\left(y_n^{(1-\eta)}\right)(1+\delta)^{\theta(1-\eta)\alpha}\left[\mathbb{E}(y_n^{\eta}) - \mathbb{E}(y_n^{\eta})(1+\delta)\right] \\ & + n_o\,\mathbb{E}\left(y_n^{(1-\eta)}\right)(1+\delta)^{\theta(1-\eta)\alpha}\left[\mathbb{E}(y_n^{\eta})(1+\delta)^{\theta\eta\alpha} - \mathbb{E}(y_n^{\eta})(1+\delta)\right].\end{aligned} \tag{A.9}$$

A.4 The one-dimensional dynamics from discrete to continuous time

Our basic model, as specified by Angeletos and La'O (2013), unfolds in discrete time and it is characterized by a specific succession of events, such as the production, the employment decisions and the actual trading. However, in order to study the dynamics of the economy, we will treat time as continuous and build a dynamic model similar to that developed by Lux (1995). In what follows we briefly show how we derive a continuous dynamics from a model originally expressed in discrete time, in the case without social influence. In the latter, the opinion index x in time $t+\varepsilon$ is given by:

$$x(t+\varepsilon) = \frac{n_n(t+\varepsilon) - n_p(t+\varepsilon)}{n}, \tag{A.10}$$

where

$$n_n(t+\varepsilon) = n_n(t) - n_n(t)p_{np}(\varepsilon, \bar{\pi}_{n-p,t}(x)) + n_p(t)p_{pn}(\varepsilon, \bar{\pi}_{n-p,t}(x))$$
$$n_p(t+\varepsilon) = n_p(t) - n_p(t)p_{pn}(\varepsilon, \bar{\pi}_{n-p,t}(x)) + n_n(t)p_{np}(\varepsilon, \bar{\pi}_{n-p,t}(x)). \tag{A.11}$$

In words, $n_n(t+\varepsilon)$ is given by the number of islands that were neutral in time t, minus those of them which became pessimistic $(n_n(t)p_{np}(\varepsilon, \bar{\pi}_{n-p,t}(x)))$, plus those which were pessimistic in time t and became neutral $(n_p(t)p_{pn}(\varepsilon, \bar{\pi}_{n-p,t}(x)))$. p_{np} and p_{pn} are the probability of switching from neutral belief to pessimism and *vice versa*, respectively. They depend on the time interval ε and on $\bar{\pi}_{n-p,t}(x)$, which is the difference between the expected profits of neutral and pessimistic islands in t. Substituting (A.11) into (A.10), we can then compute the change over time and, for $\varepsilon \to 0$, we get $\dot{x}$, i.e. the change of x in continuous time:

$$\dot{x} = \lim_{\varepsilon \to 0} \frac{x(t+\varepsilon) - x(t)}{\varepsilon} = \frac{2n_p(t)q_{pn}}{n} - \frac{2n_n(t)q_{np}}{n} \tag{A.12}$$

where $q_{np} = \lim\limits_{\varepsilon \to 0} \frac{p_{np}}{\varepsilon}$ and $q_{pn} = \lim\limits_{\varepsilon \to 0} \frac{p_{pn}}{\varepsilon}$ are the transition rates from neutral to pessimism and *vice versa* of the system in continuous time.

From equation (4.2) it follows that an example of switching probabilities satisfying $q_{np} = \lim\limits_{\varepsilon \to 0} \frac{p_{np}}{\varepsilon}$ and $q_{pn} = \lim\limits_{\varepsilon \to 0} \frac{p_{pn}}{\varepsilon}$ may be

$$p_{np} = e^{-(a_0\pi(x))}\varepsilon \qquad \text{and} \qquad p_{pn} = e^{(a_0\pi(x))}\varepsilon. \tag{A.13}$$

In fact, $\bar{\pi}_{n-p,t}(x)$ is bounded from above and, thus, also q_{np} and q_{pn}; therefore it is possible to choose an ε small enough such that p_{pn} and p_{np} are smaller or equal than one.

A.5 Welfare

Utility is given by

$$U_i = \Sigma_{t=0}^{\infty}\beta^t \left[\left(\frac{c_{it}}{1-\eta} \right)^{1-\eta} \left(\frac{c_{it}^*}{\eta} \right)^{\eta} - \frac{l_{it}^{\varepsilon}}{\varepsilon} \right]. \tag{A.14}$$

In equilibrium, $c_{it} = (1-\eta)y_{it}$ and $c_{it}^* = \eta y_{jt}$. Moreover, $l_{it} = (\mathbb{E}_{it}[p_{it}]\Theta y_{it})^{\frac{1}{\varepsilon}} = (\mathbb{E}_{it}[y_{jt}^{\eta}]\Theta y_{it}^{1-\eta})^{\frac{1}{\varepsilon}}$, so we can rewrite utility as

$$U_i = \Sigma_{t=0}^{\infty}\beta^t \left[(y_{it})^{1-\eta} \left((y_{jt})^{\eta} - \mathbb{E}_{it}[y_{jt}^{\eta}] \frac{\Theta}{\varepsilon} \right) \right], \tag{A.15}$$

where we already defined $\frac{\Theta}{\varepsilon} \equiv \theta$. By substituting expected output of the different types, we obtain equation (5.2).

A.6 Expected output and welfare

In the one-dimensional system, we compute the probability of the economy to converge to a certain stable steady state, $p_{x_k^*}$, as follows:

$$p_{x_k^*} = \frac{(x_u^* + 1)}{2}, \tag{A.16}$$

where $k \in \{1, 2, 3\}$ indicates the stable equilibrium points and $u \in \{1, 2, 3\}$, with $k \neq u$, indicates the unstable equilibrium point.

We compute expected production, or welfare, considering all steady states, as follows:

$$Y = \sum_{m=1}^{N} p_{x_k^*} \, \mathbb{E}(y_k), \tag{A.17}$$

$N \in \{1, 2\}$ is the total number of stable equilibiurm points. $\mathbb{E}(y_k)$ represents expected output or welfare at equilibrium point k.

Appendix B Proofs

B.1 Proof of proposition 1

The average output of neutral islands is given by:

$$\int_0^{+\infty} K_1^{\eta\gamma} A_i^{\eta\alpha} \, \mathbb{E}(A_i^{\eta\alpha})^{\eta\gamma\theta} dF(A_i), \tag{B.1}$$

which can be rewritten as $K_1^{\eta\gamma} \, \mathbb{E}(A_i^{\eta\alpha})^{\eta\gamma\theta+1}$. Given that $\gamma \equiv \frac{1}{1-\theta}$ and $\alpha \equiv \frac{1}{1-\theta+\eta\theta}$, we have that $\eta\gamma\theta + 1 = \frac{\gamma}{\alpha}$ and therefore the average output of neutral islands equals the neutral belief $K_1^{\eta\gamma} \, \mathbb{E}(A_i^{\eta\alpha})^{\frac{\gamma}{\alpha}}$.

B.2 Proof of Proposition 2

In an economy populated by two expectation types, neutral islands' expected profits are given by the sum of the profits earned by trading with both expectation types multiplied by their shares:

$$\begin{aligned} \bar{\pi}_{n,t}(x) = \mathbb{E}\left(y_n^{(1-\eta)}\right) \mathbb{E}(y_n^{\eta}) \frac{1-x}{2} & \left[(1-\delta)^{\theta\eta\alpha} - 1\right] \\ + \frac{x+1}{2} & \left[1 - 1\right]. \end{aligned} \tag{B.2}$$

For $x = 1$, equation (B.2) equals zero; for $-1 \leq x < 1$, given that $(1-\delta)^{\theta\eta\alpha} < 0$, it takes negative values. Analogously, pessimistic islands' expected profits are given by

$$\begin{aligned}\bar{\pi}_{p,t}(x) =& \frac{1-x}{2}(1-\delta)^{\theta(1-\eta)\alpha}\left[(1-\delta)^{\theta\eta\alpha} - (1-\delta)\right] \\ &+ \frac{x+1}{2}(1-\delta)^{\theta(1-\eta)\alpha}\left[1-(1-\delta)\right]\Bigg].\end{aligned} \tag{B.3}$$

For $x = 1$, equation (B.3) is positive; for $-1 \leq x < 1$, it takes positive values as well, for that $0 < \delta < 1$ and $0 < \theta\eta\alpha < 1 \Rightarrow (1-\delta)^{\theta\eta\alpha} > (1-\delta)$.

B.3 Proof of Proposition 3

The fixed point of equation (4.3) is given by

$$\tanh(a_0\bar{\pi}_{n-p}(x)) = x. \tag{B.4}$$

$x = 1$ represent an economy where all agents have rational expectations; however, in order for $\tanh(a_0\bar{\pi}_{n-p}(x)) = 1$, $a_0\bar{\pi}_{n-p}(x) \to \infty$, which is impossible considering that $a_0 \geq 0$ and $\bar{\pi}_{n-p}(x) < 0$.

Instead, for $a_0 \to \infty$, it is possible for $x = -1$ to be a fixed point of equation (4.3).

B.4 Proof of Proposition 4

In an economy populated by three expectation types, neutral islands' expected profits are given by the sum of the profits earned by trading with all expectation types multiplied by their shares:

$$\begin{aligned}\mathbb{E}(\pi_{n,t}) = &\left[\mathbb{E}\left(y_n^{(1-\eta)}\right)\mathbb{E}(y_n^{\eta})\right] n_p\left[(1-\delta)^{\theta\eta\alpha} - 1\right] \\ &+ n_n\left[1-1\right] \\ &+ n_o\left[(1+\delta)^{\theta\eta\alpha} - 1\right],\end{aligned} \tag{B.5}$$

from which we notice that neutral islands earn negative, zero, and positive expected profits when trading with pessimistic, neutral, and optimistic islands, respectively.

Analogously, optimistic islands' expected profits are given by

$$\begin{aligned}\mathbb{E}(\pi_{o,t}) = &\left[\mathbb{E}\left(y_n^{(1-\eta)}\right)\mathbb{E}(y_n^{\eta})\right] n_p(1+\delta)^{\theta(1-\eta)\alpha}\left[(1-\delta)^{\theta\eta\alpha} - (1+\delta)\right] \\ &+ n_n(1+\delta)^{\theta(1-\eta)\alpha}\left[1-(1+\delta)\right] \\ &+ n_o(1+\delta)^{\theta(1-\eta)\alpha}\left[(1+\delta)^{\theta\eta\alpha} - (1+\delta)\right],\end{aligned} \tag{B.6}$$

from which it emerges that optimistic islands earn negative expected profits from all matches. In particular, by comparing the negative factor in equation (B.5), $(1-\delta)^{\theta\eta\alpha} - 1$, with one of the negative factors in (B.6), $(1+\delta)^{\theta\eta\alpha} - (1+\delta)$, we observe that the former is less negative than the latter, given that $\delta > 0$.

Appendix C Robustness

C.1 Difference in expected profits of neutral and pessimistic islands, for different parameters

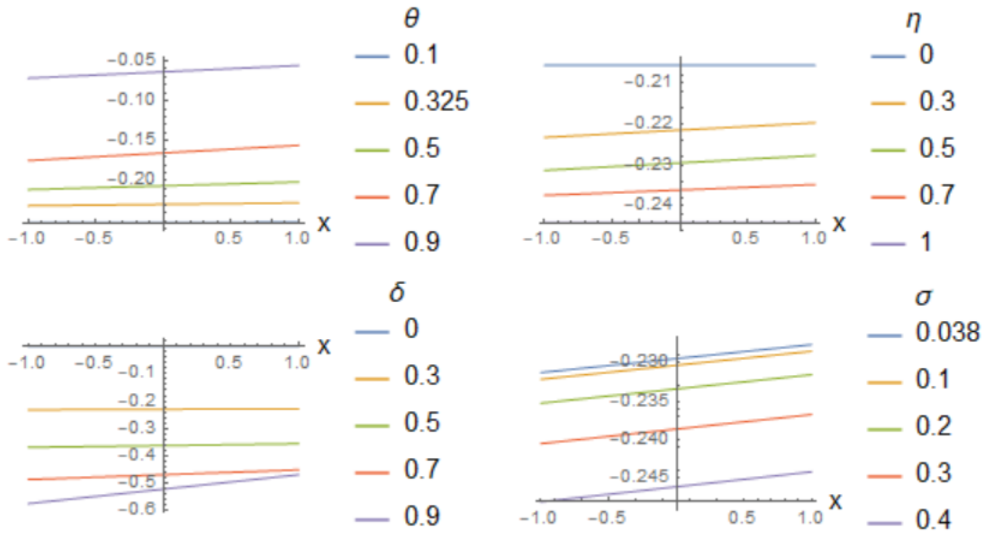

Figure C.1: Difference in expected profits of neutral and pessimistic islands, for different parameters.

From equation (4.4) and figure C.1, we observe that all the parameters have the same effect on the difference, except θ. A rise of η expands the difference of expected profits between the two groups. In fact, on the one hand, it reduces what a neutral island earns by meeting a pessimistic island, because it increases the strategic complementarity, which, in turn, diminishes pessimists' output. On the other hand, it increases both the revenues of pessimistic islands and their costs when they meet both types, but the positive effect on the former is stronger than the negative on the latter. The idea is that the higher η, the stronger is the effect of the trading partner's output on an island's price and, thus, on her terms of trade. Likewise, δ decreases pessimistic islands' output and an increment in σ increases the expected productivity of islands amplifying the difference. θ, overall, diminishes $\bar{\pi}_{n-p,t}(x)$: in fact, it reduces the profits of neutral islands meeting pessimists, but it also reduces the revenues of pessimistic islands more than it decreases their cost, regardless of the type of the trading partner.

C.2 Sentiment dynamics without social influence

Figure C.2: Dynamics of $\dot{x}$, without social influence, under different parameters.

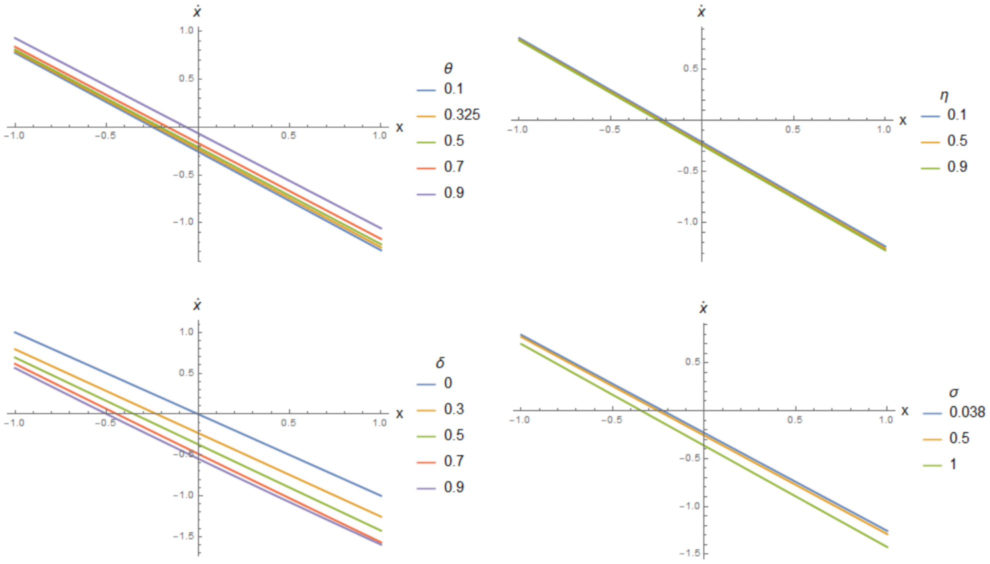

Table C.1: Equilibrium points of the dynamics shown in figure C.2.

	Equilibrium points	Derivative at the equilibrium
$\delta = 0$	$x^* = 0$	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -1
$\delta = 0.3$	x^* = -0.2259	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -1.0248
$\delta = 0.5$	x^* = -0.3488	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -1.0610
$\delta = 0.7$	x^* = -0.4431	$\frac{d(\dot{x})}{dx}\|_{x^*}$ =-1.0997
$\delta = 0.9$	x^* = -0.4985	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -1.1078
$\theta = 0.1$	x^* = -0.2471	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -1.0319
$\theta = 0.325$	x^* = -0.2259	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -1.0248
$\theta = 0.5$	x^* = -0.2043	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -1.0169
$\theta = 0.7$	x^* = -0.1652,	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -1.0045
$\theta = 0.9$	x^* = -0.0640	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -0.9941
$\eta = 0.1$	x^* = -0.2086	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -1.0216
$\eta = 0.5$	x^* = -0.2259	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -1.0248
$\eta = 0.9$	x^* = -0.2371	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -1.0289
$\sigma = 0.038$	x^* = -0.2259	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -1.0248
$\sigma = 0.5$	x^* = -0.2511	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -1.0312
$\sigma = 1$	x^* = -0.3433	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -1.0621

Figure C.3: Isoclines of the two-dimensional system without social influence and $a_0 = 0$.

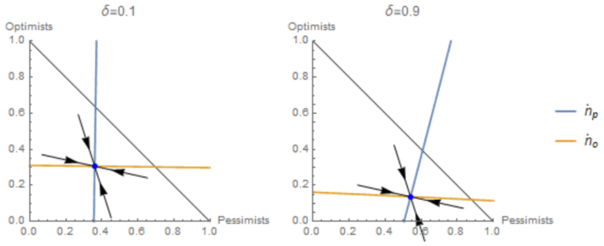

Figure C.4: Isoclines of the two-dimensional system without social influence and $a_0 = 1.5$.

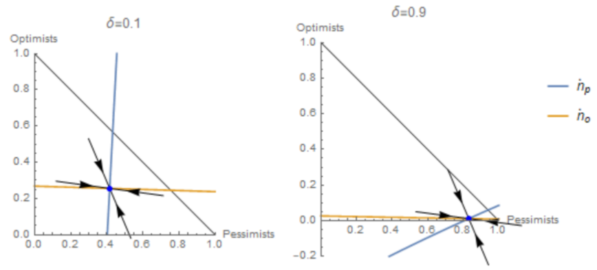

Figure C.5: Isoclines of the two-dimensional system without social influence and $a_0 = 2$.

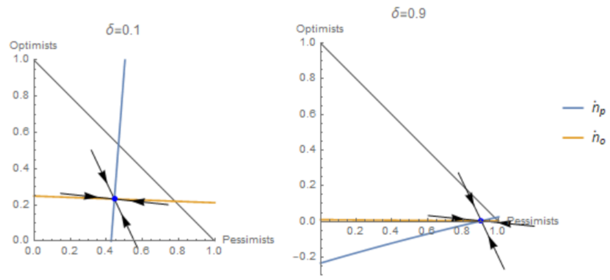

Figure C.6: Isoclines of the two-dimensional system without social influence and $a_0 = 1$.

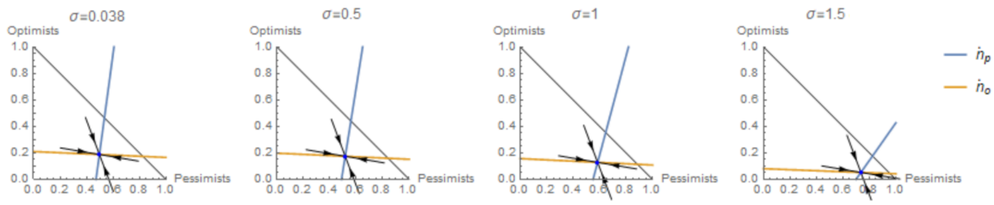

Table C.2: Equilibrium points of the dynamics shown in figure C.3, C.4, C.5 and C.6.

	Equilibrium points	Eigenvalues	Eigenvectors
$a_0 = 1, \delta = 0.1$	(n_p^*, n_o^*) = (0.3883, 0.2805)	λ_1 =-1.8124 λ_2 =-1.3612	v_1 = (-0.3523, 0.9359) v_2 = (0.9858, -0.1678)
$a_0 = 1, \delta = 0.9$	(n_p^*, n_o^*) = (0.7146, 0.0436)	λ_1 =-1.8166 λ_2 =-1.3607	v_1 = (-0.3525, 0.9358) v_2 = (0.9863, -0.1649)
$a_0 = 1.5, \delta = 0.1$	(n_p^*, n_o^*) = (0.4165, 0.2557)	λ_1 =-2.0637 λ_2 =-1.339	v_1 = (-0.3918, 0.9200) v_2 = (0.9911, -0.1333)
$a_0 = 1.5, \delta = 0.9$	(n_p^*, n_o^*) = (0.8311, 0.0119)	λ_1 =-2.0735 λ_2 =-1.3394	v_1 = (-0.3929, 0.9196) v_2 = (0.9916, -0.1294)
$a_0 = 1, \sigma = 0.038$	(n_p^*, n_o^*) = (0.4974, 0.1879)	λ_1 =-1.8139 λ_2 =-1.3610	v_1 = (-0.3524, 0.9359) v_2 = (0.9860, -0.1666)
$a_0 = 1, \sigma = 0.5$	(n_p^*, n_o^*) = (0.5163, 0.1741)	λ_1 =-1.8142 λ_2 =-1.361	v_1 = (-0.3524, 0.9359) v_2 = (0.9860, -0.1665)
$a_0 = 1, \sigma = 1$	(n_p^*, n_o^*) = (0.5850, 0.1284)	λ_1 =-1.8150 λ_2 =-1.3609	v_1 = (-0.3524, 0.9358) v_2 = (0.9861, -0.1659)

C.3 Sentiment dynamics with social influence

Figure C.7: Dynamics of $\dot{x}$ with $a_0 = 1$, $a_1 = 1.54$ and varying parameters.

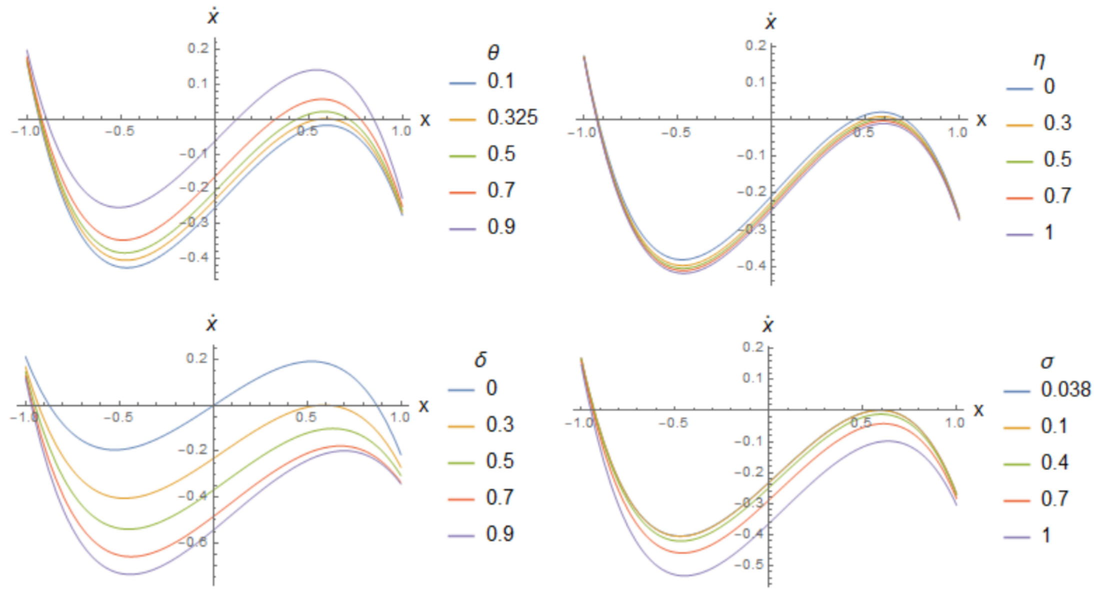

Table C.3: Equilibrium points of the dynamics shown in figure C.7.

	Equilibrium points	Derivative at the equilibrium
$a_1 = 1.5357, \theta = 0.1$	$x^* =$ -0.9343	$\frac{d(\dot{x})}{dx}\vert_{x^*} =$ -2.2562
$a_1 = 1.5357, \theta = 0.5$	$x_1^* =$ -0.9273 $x_2^* =$ 0.4423 $x_3^* =$ 0.7106	$\frac{d(\dot{x})}{dx}\vert_{x_1^*} =$ -2.0931 $\frac{d(\dot{x})}{dx}\vert_{x_2^*} =$ 0.2666 $\frac{d(\dot{x})}{dx}\vert_{x_3^*} =$ -0.3373
$a_1 = 1.5357, \theta = 0.7$	$x_1^* =$ -0.9196 $x_2^* =$ 0.3253 $x_3^* =$ 0.7747	$\frac{d(\dot{x})}{dx}\vert_{x_1^*} =$ -1.9374 $\frac{d(\dot{x})}{dx}\vert_{x_2^*} =$0.4037 $\frac{d(\dot{x})}{dx}\vert_{x_3^*} =$-0.6045
$a_1 = 1.5357, \theta = 0.9$	$x_1^* =$-0.8946 $x_2^* =$ 0.1179 $x_3^* =$ 0.8462	$\frac{d(\dot{x})}{dx}\vert_{x_1^*} =$ -1.5480 $\frac{d(\dot{x})}{dx}\vert_{x_2^*} =$ 0.5258 $\frac{d(\dot{x})}{dx}\vert_{x_3^*} =$ -1.0543

$a_1 = 1.5357, \eta = 0$	$x_1^* =$ -0.9257 $\quad \frac{d(\dot{x})}{dx}\vert_{x_1^*}$ =-2.062 $x_2^* = 0.4475 \quad \frac{d(\dot{x})}{dx}\vert_{x_2^*} = 0.2551$ $x_3^* = 0.7046 \quad \frac{d(\dot{x})}{dx}\vert_{x_3^*} =$ -0.3196
$a_1 = 1.5357, \eta = 0.3$	$x_1^* =$ -0.9288 $\quad \frac{d(\dot{x})}{dx}\vert_{x_1^*} =$ -2.1282 $x_2^* = 0.51031 \quad \frac{d(\dot{x})}{dx}\vert_{x_2^*} = 0.1594$ $x_3^* = 0.6620 \quad \frac{d(\dot{x})}{dx}\vert_{x_3^*}$ =-0.1819
$a_1 = 1.5357, \eta = 0.5$	$x^*1 =$ -0.9302 $\quad \frac{d(\dot{x})}{dx}\vert_{x_1^*} =$ -2.1593 $x_2^* = 0.5893 \quad \frac{d(\dot{x})}{dx}\vert_{x_2^*} = 0.000$
$a_1 = 1.5357, \eta = 0.7$	$x^* =$ -0.9312 $\quad \frac{d(\dot{x})}{dx}\vert_{x^*} =$ -2.1832
$a_1 = 1.5357, \eta = 1$	$x^* =$ -0.9324 $\quad \frac{d(\dot{x})}{dx}\vert_{x^*} =$ -2.2098
$a_1 = 1.5357, \delta = 0$	$x_1^* =$ -0.8711 $\quad \frac{d(\dot{x})}{dx}\vert_{x_1^*} =$ -1.2825 $x_2^* = 0 \quad \frac{d(\dot{x})}{dx}\vert_{x_2^*} = 0.5357$ $x_3^* = 0.8711 \quad \frac{d(\dot{x})}{dx}\vert_{x_3^*} =$ -1.2825
$a_1 = 1.5357, \delta = 0.3$	$x_1^* =$ -0.9302 $\quad \frac{d(\dot{x})}{dx}\vert_{x_1^*} =$ -2.1592 $x_2^* = 0.5893 \quad \frac{d(\dot{x})}{dx}\vert_{x_2^*} = 0.000$
$a_1 = 1.5357, \delta = 0.5$	$x^* =$ -0.9494 $\quad \frac{d(\dot{x})}{dx}\vert_{x^*} =$ -2.7010
$a_1 = 1.5357, \delta = 0.7$	$x^* =$ -0.9612 $\quad \frac{d(\dot{x})}{dx}\vert_{x^*} =$ -3.1963
$a_1 = 1.5357, \delta = 0.9$	$x^* =$ -1 $\quad \frac{d(\dot{x})}{dx}\vert_{x^*} =$ -90789.9
$a_1 = 1.5357, \sigma = 0.1$	$x_1^* =$ -0.9303 $\quad \frac{d(\dot{x})}{dx}\vert_{x^*} =$ -2.1626 $x_2^* = 0.5916 \quad \frac{d(\dot{x})}{dx}\vert_{x^*} = 0.0000$
$a_1 = 1.5357, \sigma = 0.4$	$x^* =$ -0.9330 $\quad \frac{d(\dot{x})}{dx}\vert_{x^*} =$ -2.2242
$a_1 = 1.5357, \sigma = 0.7$	$x^* =$ -0.9389 $\quad \frac{d(\dot{x})}{dx}\vert_{x^*} =$ -2.3765
$a_1 = 1.5357, \sigma = 1$	$x^* =$ -0.9484 $\quad \frac{d(\dot{x})}{dx}\vert_{x^*} =$ -2.6675

Figure C.8: Dynamics of $\dot{x}$ with $a_0 = 1$, $a_1 = 1.75$ and varying v and δ.

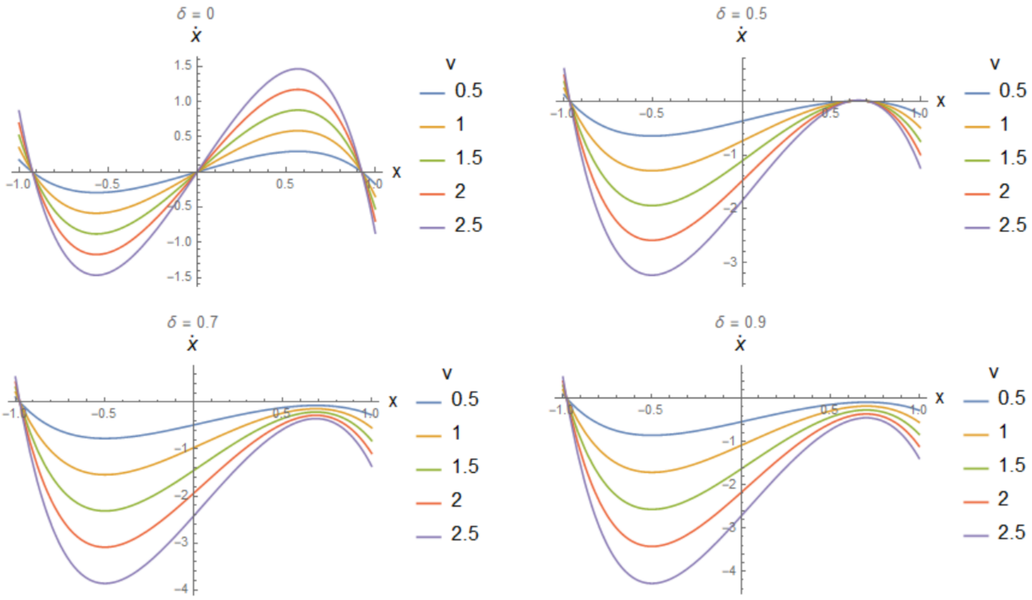

Table C.4: Equilibrium points of the dynamics shown in figure C.8.

	Equilibrium points	Derivative at the equilibrium
$a_1 = 1.74299, \delta = 0, v = 1$	$x_1^* =$ -0.9230 $x_2^* = 0$ $x_3^* = 0.9230$	$\frac{d(\dot{x})}{dx}\|_{x_1^*} =$ -3.8550 $\frac{d(\dot{x})}{dx}\|_{x_2^*} = 1.4860$ $\frac{d(\dot{x})}{dx}\|_{x_3^*} =$ -3.8550
$a_1 = 1.74299, \delta = 0, v = 1.5$	" " "	$\frac{d(\dot{x})}{dx}\|_{x_1^*} =$ -5.7825 $\frac{d(\dot{x})}{dx}|_{x_2^*} =$2.2290 $\frac{d(\dot{x})}{dx}\|_{x_3^*} =$ -5.7825
$a_1 = 1.74299, \delta = 0, v = 2$	" " "	$\frac{d(\dot{x})}{dx}\|_{x_1^*} =$ -7.7100 $\frac{d(\dot{x})}{dx}\|_{x_2^*} = 2.9720$ $\frac{d(\dot{x})}{dx}\|_{x_3^*} =$ -7.7100
$a_1 = 1.74299, \delta = 0, v = 2.5$	" " "	$\frac{d(\dot{x})}{dx}\|_{x_1^*} =$ -9.6375 $\frac{d(\dot{x})}{dx}\|_{x_2^*} = 3.7150$ $\frac{d(\dot{x})}{dx}\|_{x_3^*} =$ -9.6375

$a_1 = 1.74299, \delta = 0.5, v = 1$	$x_1^* =$ -0.9677 $x_2^* =$ 0.6545	$\frac{d(\dot{x})}{dx}\|_{x_1^*}$ =-7.0514 $\frac{d(\dot{x})}{dx}\|_{x_2^*}$ =0.0000
$a_1 = 1.74299, \delta = 0.5, v = 1.5$	" "	$\frac{d(\dot{x})}{dx}\|_{x_1^*}$ = -10.577 $\frac{d(\dot{x})}{dx}\|_{x_2^*}$ = 0.0000
$a_1 = 1.74299, \delta = 0.5, v = 2$	" "	$\frac{d(\dot{x})}{dx}\|_{x_1^*}$ =-14.1027 $\frac{d(\dot{x})}{dx}\|_{x_2^*}$ = 0.0000
$a_1 = 1.74299, \delta = 0.5, v = 2.5$	" "	$\frac{d(\dot{x})}{dx}\|_{x_1^*}$ =-17.6284 $\frac{d(\dot{x})}{dx}\|_{x_2^*}$ = 0.0000
$a_1 = 1.74299, \delta = 0.7, v = 1$	$x^* =$ -0.9750	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -8.2204
$a_1 = 1.74299, \delta = 0.7, v = 1.5$	"	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -12.3305
$a_1 = 1.74299, \delta = 0.7, v = 2$	"	$\frac{d(\dot{x})}{dx}\|_{x^*}$ =-16.4407
$a_1 = 1.74299, \delta = 0.7, v = 2.5$	"	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -20.5509
$a_1 = 1.74299, \delta = 0.9, v = 1$	$x^* =$ -0.9793	$\frac{d(\dot{x})}{dx}\|_{x_1^*}$ = -9.1469
$a_1 = 1.74299, \delta = 0.9, v = 1.5$	"	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -13.8159
$a_1 = 1.74299, \delta = 0.9, v = 2$	"	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -18.4212
$a_1 = 2, \delta = 0.9, v = 2.5$	"	$\frac{d(\dot{x})}{dx}\|_{x^*}$ = -23.0266

Figure C.9: Bifurcation diagram for the parameters shown in table 4.1, for varying a_0

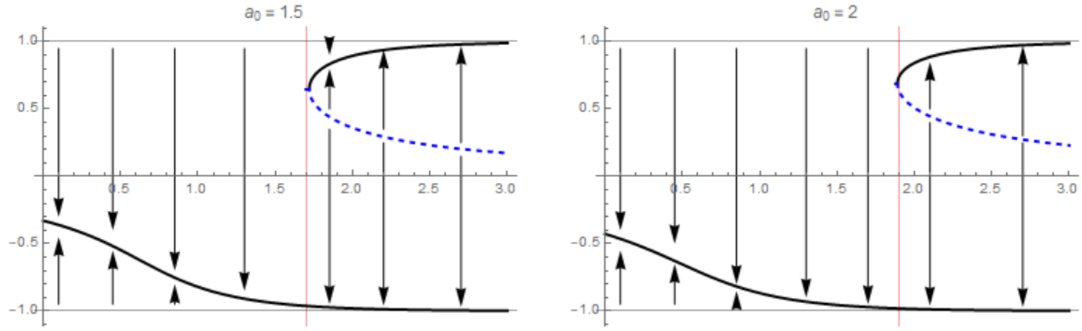

Figure C.10: Bifurcation diagram for the parameters shown in table 4.1, $a_0 = 1$ and for varying δ

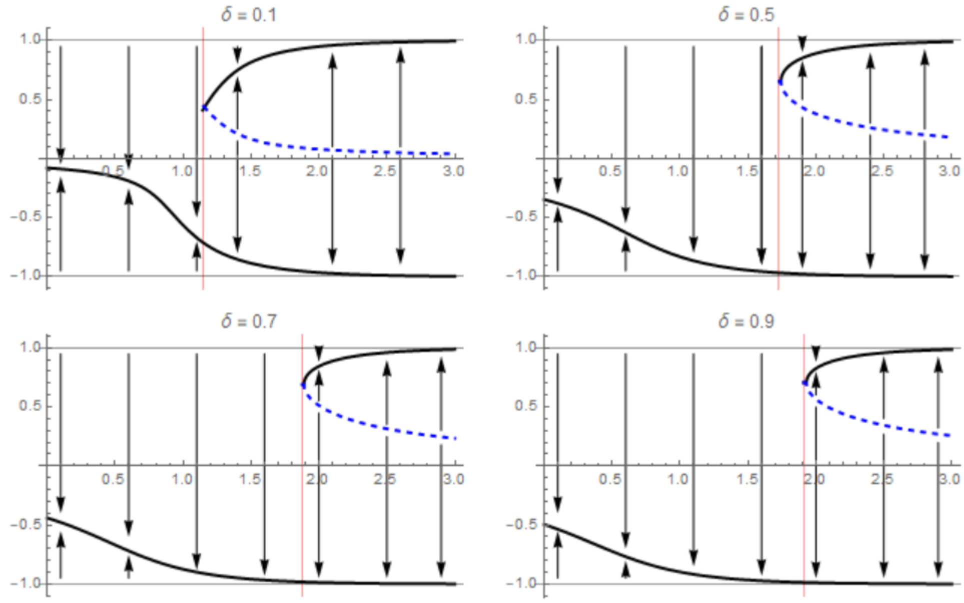

Figure C.11: Bifurcation diagram for the parameters shown in table 4.1, $a_0 = 1$ and for varying η

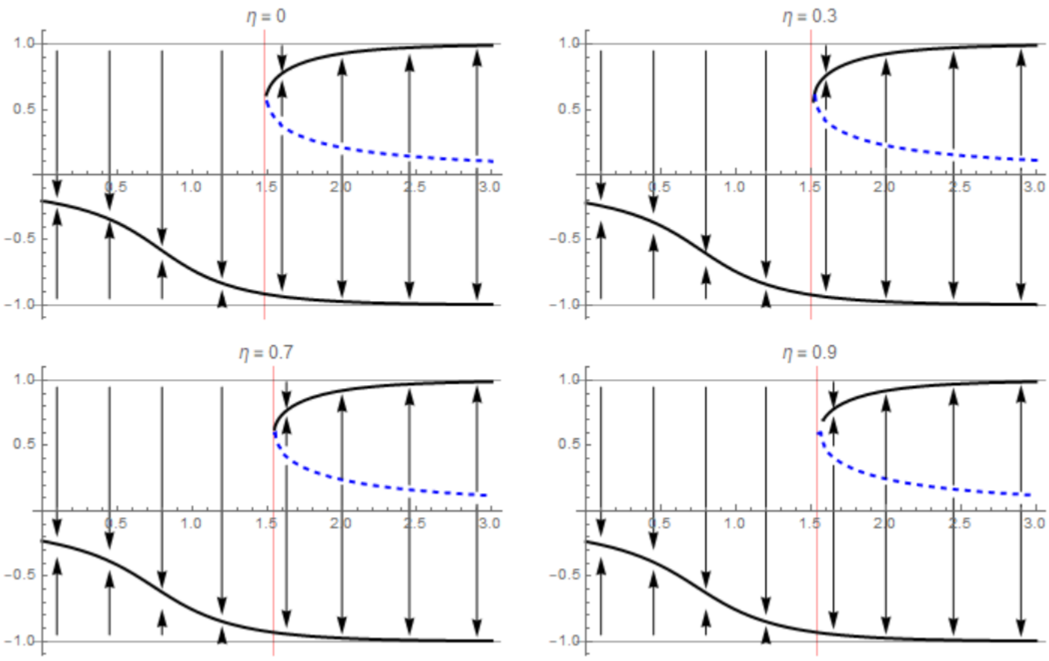

Figure C.12: Bifurcation diagram for the parameters shown in table 4.1, $a_0 = 1$ and for varying θ

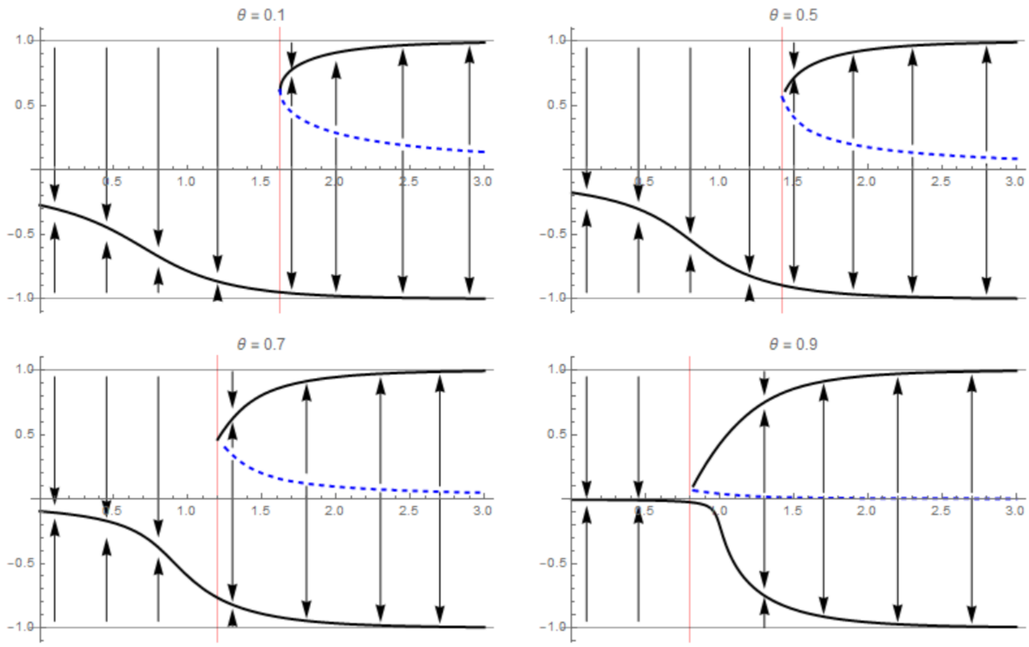

Figure C.13: First bifurcation point with different a_0.

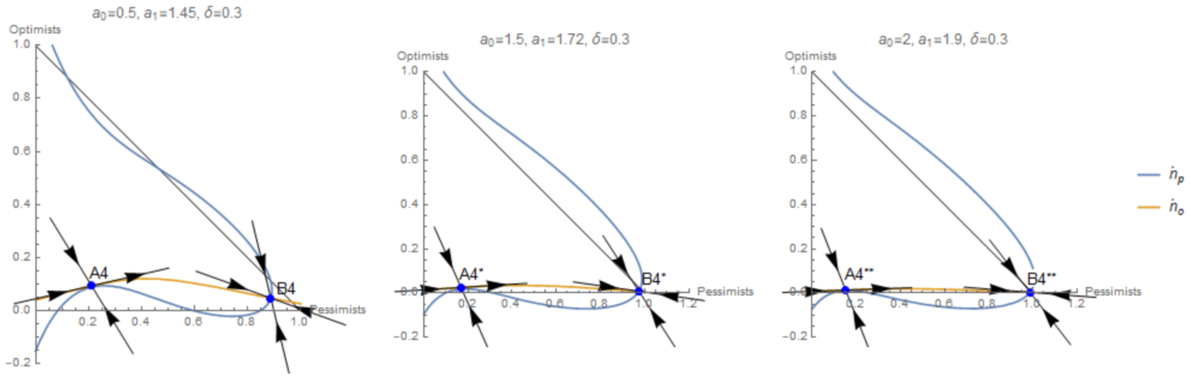

Table C.5: Equilibrium points shown in figure C.13.

	Equilibrium points	Eigenvalues	Eigenvectors
$a_0 = 0.5, a_1 = 1.45$	$A4 = (0.2115,\ 0.0937)$	$\lambda_1 = -1.5454$	$v_1 =$ (-0.5219, 0.8530)
		$\lambda_2 = -8.3188 * 10^{-}10$	$v_2 =$ (0.9980, 0.0636)
	$B4 = (0.8876,\ 0.0476)$	$\lambda_1 = -2.5828$	$v_1 =$ (-0.5436, 0.8394)
		$\lambda_2 = -1.2349$	$v_2 =$ (0.9900, -0.1409)
$a_0 = 1.5, a_1 = 1.72$	$A4^* = (0.1687,\ 0.0251)$	$\lambda_1 = -3.7317$	$v_1 =$ (-0.4057, 0.9140)
		$\lambda_2 = -6.9333 * 10^{-}8$	$v_2 =$ (0.9745, 0.2243)
	$B4^* = (0.9738,\ 0.0081)$	$\lambda_1 = -6.1116$	$v_1 =$ (-0.2357, 0.9718)
		$\lambda_2 = -3.4330$	$v_2 =$ (0.9410, -0.3383)
$a_0 = 2, a_1 = 1.9$	$A4^{**} = (0.1687,\ 0.0251)$	$\lambda_1 = -5.3593$	$v_1 =$ (-0.3684, 0.9297)
		$\lambda_2 = 5.9396 * 10^{-}9$	$v_2 =$ (0.9993, 0.0370)
	$B4^{**} = (0.9738,\ 0.0081)$	$\lambda_1 = -9.3830$	$v_1 =$ (-0.6124, 0.7905)
		$\lambda_2 = -4.9689$	$v_2 =$ (0.9970, -0.0772)

Figure C.14: First bifurcation point with $a_0 = 1$ and different δ.

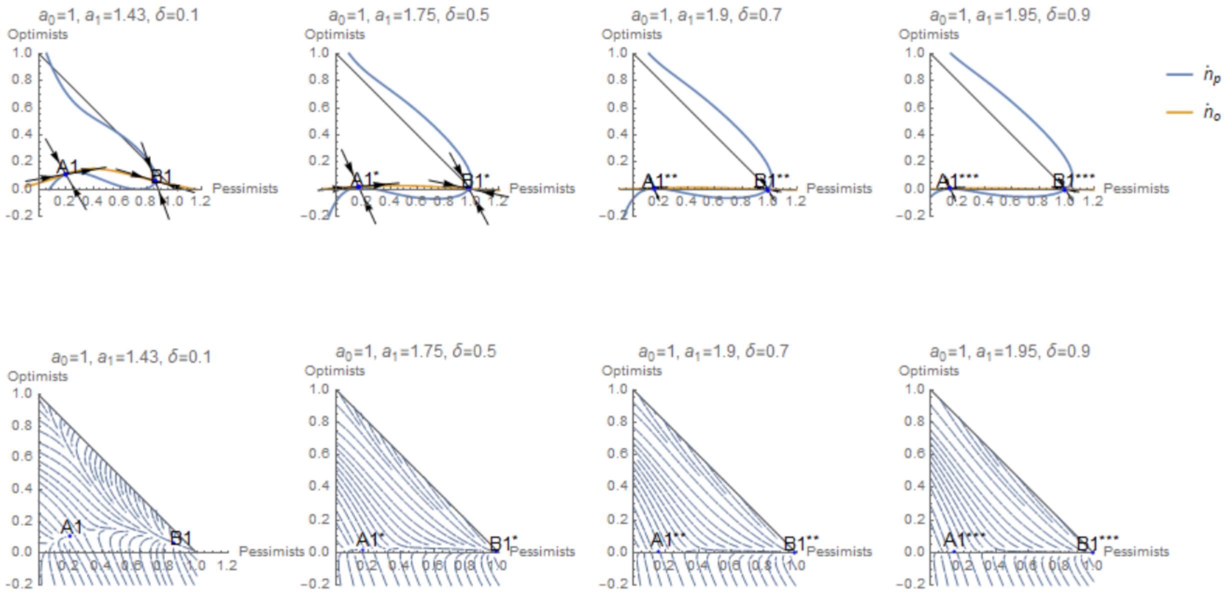

Table C.6: Equilibrium points shown in figure C.14.

	Equilibrium points	Eigenvalues	Eigenvectors
$a_1 = 1.425, \delta = 0.1$	$A1 = $ (0.2159, 0.1185)	$\lambda_1 = -1.2659$	$v_1 =$ (-0.5587, 0.8294)
		$\lambda_2 = 6.4310 * 10^-9$	$v_2 =$ (0.9532, 0.3024)
	$B1 = $ (0.8550, 0.0649)	$\lambda_1 = -2.1906$	$v_1 =$ (-0.1590, 0.9873)
		$\lambda_2 = -0.9313$	$v_2 =$ (0.9586, -0.2848)
$a_1 = 1.75, \delta = 0.5$	$A1^* = $ (0.1663, 0.0209)	$\lambda_1 = -4.1531$	$v_1 =$ (-0.3937, 0.9192)
		$\lambda_2 = 1.07604 * 10^-8$	$v_2 =$ (0.9986, 0.0532)
	$B1^* = $ (0.9774, 0.0064)	$\lambda_1 = -6.8464$	$v_1 =$ (-0.5692, 0.8222)
		$\lambda_2 = -3.6812$	$v_2 =$ (0.9934, -0.1143)
$a_1 = 1.9, \delta = 0.7$	$A1^{**} = $ (0.1519, 0.0101)	$\lambda_1 = -6.1810$	$v_1 =$ (-0.3556, 0.9346)
		$\lambda_2 = -4.03597 * 10^-9$	$v_2 =$ (0.9996, 0.0278)
	$B1^{**} =$(0.9885, 0.0024)	$\lambda_1 = -5.9684$	$v_1 =$ (-0.5045, 0.8634)
		$\lambda_2 = -3.9312$	$v_2 =$ (0.9823, -0.1874)
$a_1 = 1.9367, \delta = 0.9$	$A1^{***} = $ (0.1461, 0.0062)	$\lambda_1 = -8.0787$	$v_1 =$ (-0.3344, 0.9424)
		$\lambda_2 = 2.04594 * 10^-9$	$v_2 =$ (0.9999, 0.0167)
	$B1^{***} = $ (0.9918, 0.0012)	$\lambda_1 = -15.4894$	$v_1 =$ (-0.6527, 0.7576)
		$\lambda_2 = -5.8505$	$v_2 =$ (0.9997, -0.0260)

Figure C.15: Second bifurcation point with $a_0 = 1$ and different δ.

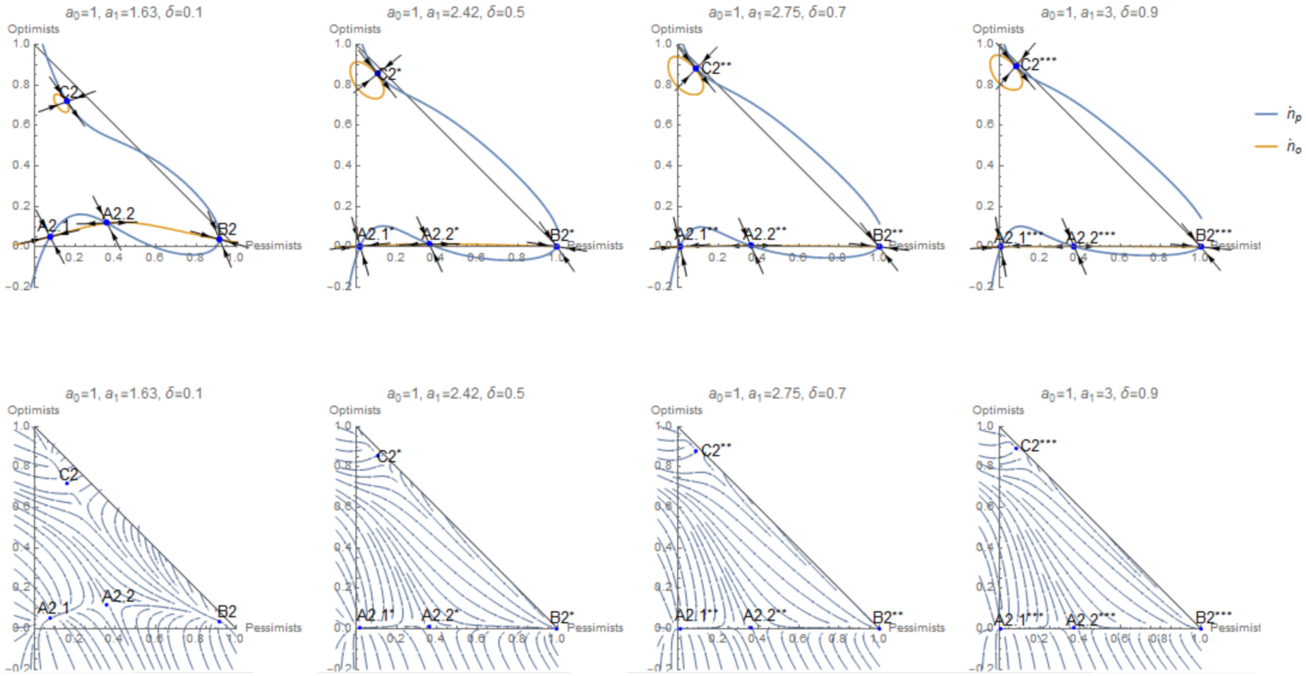

Table C.9: Equilibrium points shown in figure C.15

	Equilibrium points	Eigenvalues	Eigenvectors
$a_1 = 1.63, \delta = 0.1$	$A2.1 = (0.0762,\ 0.0507)$	$\lambda_1 = -2.4010$	$v_1 =$ (-0.5822, 0.8130)
		$\lambda_2 = -0.9268$	$v_2 =$ (0.9006, 0.4346)
	$A2.2 = (0.3581,\ 0.1184)$	$\lambda_1 = -1.0764$	$v_1 =$ (-0.4779, 0.8784)
		$\lambda_2 = 0.5069$	$v_2 =$ (0.9969, 0.0786)
	$B2 = (0.9167,\ 0.0377)$	$\lambda_1 = -2.9281$	$v_1 =$ (-0.1747, 0.9846)
		$\lambda_2 = -1.5874$	$v_2 =$ (0.9262, -0.3770)
	$C2 = (0.1604,\ 0.7195)$	$\lambda_1 = -1.2131$	$v_1 =$ (-0.9754, -0.2206)
		$\lambda_2 = 8.2542 * 10^-10$	$v_2 =$ (0.5333, -0.8459)
$a_1 = 2.42, \delta = 0.5$	$A2.1^* = (0.0193,\ 0.0036)$	$\lambda_1 = -9.1308$	$v_1 =$ (-0.2269, 0.9739)
		$\lambda_2 = -3.0854$	$v_2 =$ (0.9987, 0.052)
	$A2.2^* = (0.3630,\ 0.0134)$	$\lambda_1 = -5.5033$	$v_1 =$ (-0.4229, 0.9062)
		$\lambda_2 = 1.3804$	$v_2 =$ (0.9999, 0.0107)
	$B2^* = (0.9945,\ 0.0016)$	$\lambda_1 = -13.1787$	$v_1 =$ (-0.6370, 0.7709)
		$\lambda_2 = -7.9136$	$v_2 =$ (0.9976, -0.0694)
	$C2^* = (0.1051,\ 0.8564)$	$\lambda_1 = -2.5105$	$v_1 =$ (-0.7720, -0.6356)
		$\lambda_2 = -1.10261 * 10^-8$	$v_2 =$ (0.6360, -0.7717)

Table C.7: Equilibrium points of the dynamics shown in figure 4.5 and 4.6.

	Equilibrium points	Eigenvalues	Eigenvectors
$a_1 = 1$	$(n_p^*, n_o^*) = (0.7966,\ 0.070)$	λ_1 =-2.1763	v_1 = (-0.3470, 0.9379)
		λ_2 =-0.8154	v_2 = (0.9728, -0.2316)
$a_1 = 1.6$	$A = (0.1873,\ 0.0455)$	λ_1 = -2.5841	v_1 = (-0.4486, 0.8937)
		$\lambda_2 = 3.8638 * 10^-9$	v_2 = (0.9938, 0.1110)
	$B = (0.9503{,}0.0180)$	λ_1 = -4.1543	v_1 = (-0.4277, 0.9039)
		$\lambda_2 = -2.36097$	v_2 = (0.9728, -0.2314)
$a_1 = 2.02$	$A1 = (0.0376,\ 0.0129)$	λ_1 = -4.8682	v_1 = (-0.3659, 0.9306)
		λ_2 = -1.9329	v_2 = (0.9891, 0.1475)
	$A2 = (0.3624,\ 0.0367)$	λ_1 = -2.9309	v_1 = (-0.4396, 0.8982)
		λ_2 = 0.9791	v_2 = (0.9997, 0.0263)
	$B^* = (0.9804,\ 0.0072)$	λ_1 = -6.3727	v_1 = (-0.5066, 0.8622)
		λ_2 = -4.2244	v_2 = (0.9804, -0.1972)
$a_1 = 2.05$	$A1^* = (0.0348,\ 0.0120)$	λ_1 = -5.0352	v_1 = (-0.3594, 0.9332)
		λ_2 = -2.0562	v_2 = (0.9893, 0.1456)
	$A2^* = (0.3669,\ 0.0357)$	λ_1 = -2.9865	v_1 = (-0.4394, 0.8983)
		λ_2 = 1.0183	v_2 = (0.9997, 0.0246)
	$B^{**} = (0.9817,\ 0.0068)$	λ_1 = -6.5668	v_1 = (-0.5119, 0.8590)
		λ_2 = -4.3856	v_2 = (0.9809, -0.1943)
	$C = (0.1256,\ 0.8118)$	λ_1 = -1.8605	v_1 = (-0.8720, -0.4895)
		λ_2 = -0.00001	v_2 = (0.6042, -0.7968)
$a_1 = 2.5$	$A1^{**} = (0.0120,\ 0.0044)$	λ_1 = -8.0997	v_1 = (-0.2613, 0.9652)
		λ_2 = -4.1996	v_2 = (0.9941, 0.1089)
	$A2^{**} = (0.4109,\ 0.0222)$	λ_1 = -4.0015	v_1 = (-0.4376, 0.8991)
		λ_2 = 1.5534	v_2 = (0.9999, 0.0105)
	$B^{***} = (0.9929,\ 0.0026)$	λ_1 = -10.2595	v_1 = (-0.5818, 0.8133)
		λ_2 = -7.3972	v_2 = (0.9891, -0.1475)
	$C1 = (0.0228,\ 0.9633)$	λ_1 = -4.5529	v_1 = (-0.7717, -0.6360)
		λ_2 = -2.6134	v_2 = (0.5939, -0.8045)
	$C2 = (0.2976,\ 0.6457)$	λ_1 = -1.8165	v_1 = (-0.7334, -0.6798)
		λ_2 = 1.3444	v_2 = (0.6816, -0.7317)

Table C.8: Equilibrium points of the dynamics shown in figure 4.7.

$a_1 = 3.12$	$A1^{***} = (0.0032, 0.0012)$	$\lambda_1 =$ -15.0544,	$v_1 =$ (-0.1510, 0.9885)
		$\lambda_2 =$ -8.7216	$v_2 =$ (0.9981, 0.0621)
	$A2^{***} = (0.4398, 0.0115)$	$\lambda_1 =$ -5.9279	$v_1 =$ (-0.4378, 0.8991)
		$\lambda_2 =$ 2.2188	$v_2 =$ (0.9999, 0.0041)
	$B^{****} = (0.9981, 0.0007)$	$\lambda_1 =$ -18.9035	$v_1 =$ (-0.6429, 0.7660)
		$\lambda_2 =$ -14.2323	$v_2 =$ (0.9959, -0.0894)
	$C1^{*} = (0.3693, 0.6029)$	$\lambda_1 =$ -3.1885,	$v_1 =$ (0.6921, 0.7218)
		$\lambda_2 =$ 2.1569	$v_2 =$ (0.6993, -0.7148)
	$C2^{*} = (0.0055, 0.9910)$	$\lambda_1 =$ -8.9162	$v_1 =$ (0.4885, 0.8726)
		$\lambda_2 =$ -6.4661	$v_2 =$ (0.6284, -0.7779)
	$D = (0.1220, 0.5039)$	$\lambda_1 =$ 2.1603	$v_1 =$ (0.0664, -0.9978)
		$\lambda_2 = -7.7107 * 10^{-}9$	$v_2 =$ (-0.9102, 0.4142)
$a_1 = 4$	$A1^{****} = (0.0005, 0.0002)$	$\lambda_1 =$ -35.8835,	$v_1 =$ (-0.0631, 0.9980)
		$\lambda_2 =$ -21.7414	$v_2 =$ (0.9997, 0.0252)
	$A2^{****} = (0.4594, 0.0046)$	$\lambda_1 =$ -9.9205	$v_1 =$ (-0.4387, 0.8986)
		$\lambda_2 =$ 3.1111	$v_2 =$ (0.9999, 0.0014)
	$B^{*****} = (0.9997, 0.0001)$	$\lambda_1 =$ -45.1062	$v_1 =$ (-0.6824, 0.7310)
		$\lambda_2 =$ -34.5745	$v_2 =$ (0.9992, -0.0388)
	$C1^{**} = (0.4136, 0.5760)$	$\lambda_1 =$ -6.0773	$v_1 =$ (0.6817, 0.7316)
		$\lambda_2 =$ 3.1089	$v_2 =$ (0.7048, -0.7094)
	$C2^{**} = (0.0009, 0.9985)$	$\lambda_1 =$ -21.5163	$v_1 =$ (0.1850, 0.9827)
		$\lambda_2 =$ -16.8421	$v_2 =$ (0.6713, -0.7412)
	$D1 = (0.0241, 0.5309)$	$\lambda_1 =$ -3.2997	$v_1 =$ (-0.8932, 0.4496)
		$\lambda_2 =$ 3.2076	$v_2 =$ (0.0075, -0.9999)
	$D2 = (0.2334, 0.4310)$	$\lambda_1 =$ 2.8690	$v_1 =$ (0.1869, -0.9824)
		$\lambda_2 =$ 1.8837	$v_2 =$ (-0.9457, 0.3251)
$a_1 = 20$	$A1^{*****} = (0., 0.)$	$\lambda_1 = -3.1358 * 10^8$	$v_1 = (-6.7415 * 10^{-}9, 1.)$
		$\lambda_2 = -1.9319 * 10^8$	$v_2 = (1., 2.5589 * 10^{-}9)$
	$A2^{*****} = (0.4940, 4.889 * 10^{-}10)$	$\lambda_1 =$ -31979.3	$v_1 =$ (-0.4451, 0.8955)
		$\lambda_2 =$ 19.0006	$v_2 = (1., 1.1937 * 10^{-}10)$
	$B^{******} = (1., 0.)$	$\lambda_1 = -3.9630 * 10^8$	$v_1 =$ (-0.7071, 0.7071)
		$\lambda_2 = -3.0571 * 10^8$	$v_2 = (1., -4.2576 * 10^{-}9)$
	$C1^{***} = (0.4872, 0.5128)$	$\lambda_1 =$-21711.3	$v_1 =$ (0.6980, 0.7161)
		$\lambda_2 =$ 18.9994	$v_2 =$ (0.7071, -0.7071)
	$C2^{***} = (0., 1.)$	$\lambda_1 = -1.8808 * 10^8$	$v_1 = (1.6533 * 10^{-}8, 1.)$
		$\lambda_2 = -1.5021 * 10^8$	$v_2 =$ (0.7071, -0.7071)
	$D1^{*} = (2.0918 * 10^{-}9, 0.5067)$	$\lambda_1 =$-15460.2	$v_1 =$ (-0.8932, 0.4496)
		$\lambda_2 =$ 18.9998	$v_2 = (6.0673 * 10^{-}10, -1.)$
	$D2^{*} = (0.3205, 0.3467)$	$\lambda_1 =$ 18.738	$v_1 =$ (0.2702, -0.9628)
		$\lambda_2 =$ 18.2512	$v_2 =$ (-0.9690, 0.2471)

Table C.10: Equilibrium points of the dynamics shown in figure C.16.

	Equilibrium points	Eigenvalues	Eigenvectors
$a_1 = 2.75, \delta = 0.7$	$A2.1^{**} = (0.0113, 0.0012)$	$\lambda_1 = -15.523$	$v_1 = $ (-0.1532, 0.9882)
		$\lambda_2 = -4.2509$	$v_2 = $ (0.9998, 0.0206)
	$A2.2^{**} = (0.3624, 0.0056)$	$\lambda_1 = -8.9222$	$v_1 = $ (-0.4136, 0.9105)
		$\lambda_2 = 1.6978$	$v_2 = $ (0.9999, 0.0049)
	$B2^{**} = (0.9980, 0.0004)$	$\lambda_1 = -25.0545$	$v_1 = $ (-0.6752, 0.7376)
		$\lambda_2 = -12.6226$	$v_2 = $ (0.9997, -0.0240)
	$C2^{**} = (0.0909, 0.8818)$	$\lambda_1 = -3.0653$	$v_1 = $ (-0.7045, -0.7097)
		$\lambda_2 = -6.6986 * 10^-9$	$v_2 = $ (0.6506, -0.7594)
$a_1 = 3, \delta = 0.9$	$A2.1^{***} = (0.0068, 0.0005)$	$\lambda_1 = -24.2385$	$v_1 = $ (-0.1105, 0.9939)
		$\lambda_2 = -5.6683$	$v_2 = $ (0.9999, 0.0098)
	$A2.2^{***} = (0.3702, 0.0027)$	$\lambda_1 = -13.0734$	$v_1 = $ (-0.4098, 0.9122)
		$\lambda_2 = 1.9785$	$v_2 = $ (0.9999, 0.0022)
	$B2^{***} = (0.9991, 0.0001)$	$\lambda_1 = -43.4478$	$v_1 = $ (-0.6890, 0.7248)
		$\lambda_2 = -17.7698$	$v_2 = $ (0.9999, -0.0093)
	$C2^{***} = (0.0811, 0.8948)$	$\lambda_1 = -3.2358$	$v_1 = $ (-0.6899, -0.7239)
		$\lambda_2 = 0.0000$	$v_2 = $ (0.6509, -0.7591)

Figure C.16: Third bifurcation point with $a_0 = 1$ and different δ.

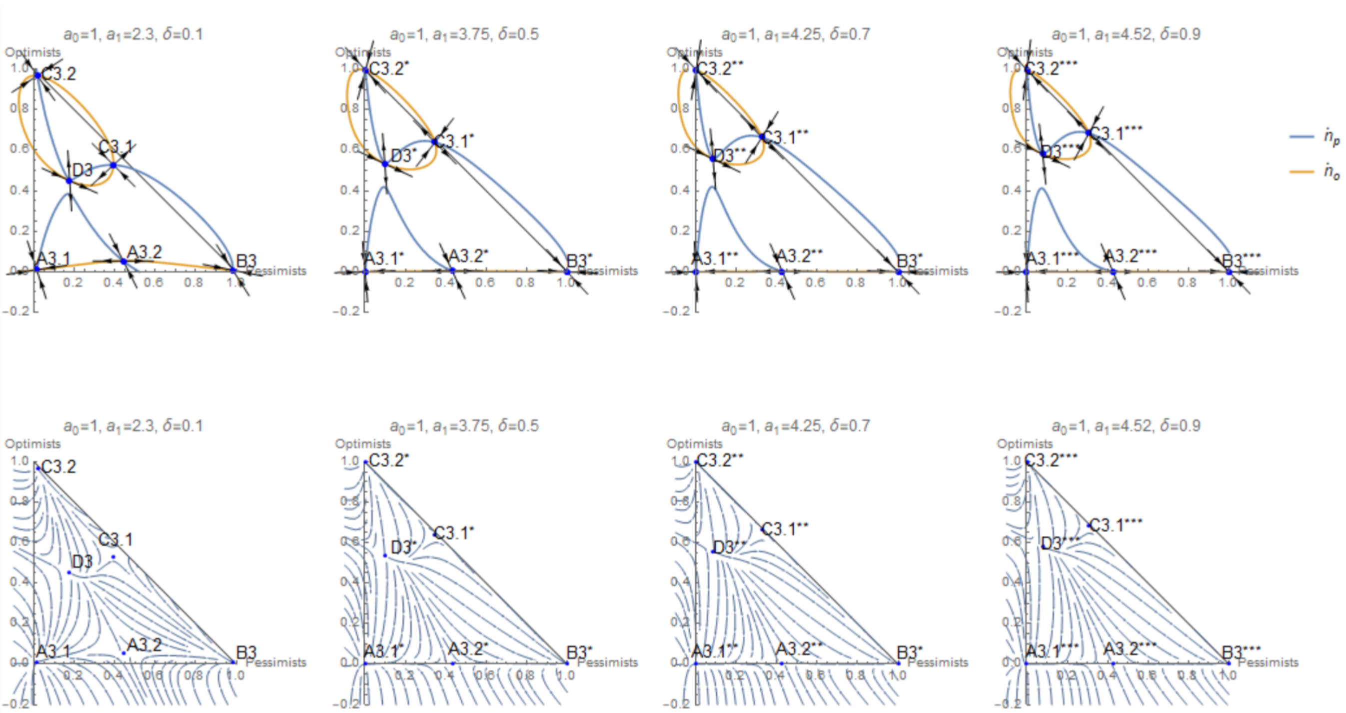

Table C.11: Equilibrium points of the dynamics shown in figure C.16.

	Equilibrium points	Eigenvalues	Eigenvectors
$a_1 = 2.3, \delta = 0.1$	$A3.1 = (0.0136,\ 0.0097)$	$\lambda_1 = -5.5142$	$v_1 =$ (-0.4939, 0.8695)
		$\lambda_2 = -3.8294$	$v_2 =$ (0.9228, 0.3853)
	$A3.2 = (0.4411,\ 0.0536)$	$\lambda_1 = -2.0404$	$v_1 =$ (-0.4482, 0.8939)
		$\lambda_2 = 1.3671$	$v_2 =$ (0.9999, 0.0107)
	$B3 = (0.9825,\ 0.0080)$	$\lambda_1 = -6.116$	$v_1 =$ (-0.2870, 0.9579)
		$\lambda_2 = -4.7003$	$v_2 =$ (0.9364, -0.3508)
	$C3.2 = (0.0168,\ 0.9691)$	$\lambda_1 = -4.6590$	$v_1 =$ (-0.9587, -0.2843)
		$\lambda_2 = -3.2160$	$v_2 =$ (0.5155, -0.8569)
	$C3.1 = (0.3931,\ 0.5278)$	$\lambda_1 = 1.3155$	$v_1 =$ (0.6942, -0.7198)
		$\lambda_2 = -1.3045$	$v_2 =$ (-0.7291, -0.6844)
	$D3 = (0.1764,\ 0.4500)$	$\lambda_1 = 1.1772$	$v_1 =$ (0.0724, -0.9974)
		$\lambda_2 = 1.94067 * 10^{-8}$	$v_2 =$ (-0.9118, 0.4106)
$a_1 = 3.75, \delta = 0.5$	$A3.1^* = (0.0011,\ 0.0002)$	$\lambda_1 = -33.8293$	$v_1 =$ (-0.0638, 0.9980)
		$\lambda_2 = -14.7287$	$v_2 =$ (0.9999, 0.0143)
	$A3.2^* = (0.4324,\ 0.0035)$	$\lambda_1 = -11.5348$	$v_1 =$ (-0.4310, 0.9024)
		$\lambda_2 = 2.8149$	$v_2 =$ (0.9999, 0.0017)
	$B3^* = (0.9996,\ 0.0001)$	$\lambda_1 = -48.7451$	$v_1 =$ (-0.6899, 0.7238)
		$\lambda_2 = -30.8416$	$v_2 =$ (0.9998, -0.0193)
	$C3.2^* = (0.0028,\ 0.9959)$	$\lambda_1 = -14.0214$	$v_1 =$ (0.2308, 0.9730)
		$\lambda_2 = -9.3672$	$v_2 =$ (0.672, -0.7405)
	$C3.1^* = (0.3462,\ 0.6397)$	$\lambda_1 = -4.9976$	$v_1 =$ (0.6670, 0.7451)
		$\lambda_2 = 2.7180$	$v_2 =$ (0.7016, -0.7126)
	$D3^* = (0.0986,\ 0.5340)$	$\lambda_1 = 2.9339$	$v_1 =$ (0.0661, -0.9978)
		$\lambda_2 = -0.4406$	$v_2 =$ (-0.9095, 0.4157)
$a_1 = 4.25, \delta = 0.7$	$A3.1^{**} = (0.0005,\ 0.0001)$	$\lambda_1 = -67.129$	$v_1 =$ (-0.0347, 0.9994)
		$\lambda_2 = -22.2154$	$v_2 =$ (0.9999, 0.0044)
	$A3.2^{**} = (0.4275,\ 0.0013)$	$\lambda_1 = -19.6166$	$v_1 =$ (-0.4265, 0.9045)
		$\lambda_2 = 3.2881$	$v_2 =$ (1., 0.0008)
	$B3^{**} = (0.9999,\ 0.0000)$	$\lambda_1 = -109.956$	$v_1 =$ (-0.7004, 0.7138)
		$\lambda_2 = -57.1282$	$v_2 =$ (0.9999, -0.0052)

	$C3.2{**} = (0.0017,\ 0.9975)$	$\lambda_1 = -19.1844$	$v_1 = (0.1460,\ 0.9893)$
		$\lambda_2 = -11.9323$	$v_2 = (0.6868,\ \text{-}0.7268)$
	$C3.1^{**} = (0.3266,\ 0.6644)$	$\lambda_1 = -6.5715$	$v_1 = (0.6482,\ 0.7615)$
		$\lambda_2 = 3.1399$	$v_2 = (0.7027,\ \text{-}0.7115)$
	$D3^{**} = (0.0853,\ 0.5573)$	$\lambda_1 = 3.3990$	$v_1 = (0.0690,\ \text{-}0.9976)$
		$\lambda_2 = 5.17766 * 10^-8$	$v_2 = (\text{-}0.9099,\ 0.4148)$
$a_1 = 4.52, \delta = 0.9$	$A3.1^{***} = (0.0003{,}0.0000)$	$\lambda_1 = -106.823$	$v_1 = (\text{-}0.0243,\ 0.9997)$
		$\lambda_2 = -28.6499$	$v_2 = (0.9999,\ 0.0020)$
	$A3.2^{***} = (0.4266,\ 0.0006)$	$\lambda_1 = -28.5918$	$v_1 = (\text{-}0.4241,\ 0.9056)$
		$\lambda_2 = 3.5711$	$v_2 = (1.,\ 0.0003)$
	$B3^{***} = (0.9999,\ 6.6381 * 10^-6)$	$\lambda_1 = -195.232$	$v_1 = (\text{-}0.7033,\ 0.7109)$
		$\lambda_2 = -81.6238$	$v_2 = (0.9999,\ \text{-}0.0019)$
	$C3.2^{***} = (0.0015,\ 0.9979)$	$\lambda_1 = -20.6881$	$v_1 = (0.1353,\ 0.9908)$
		$\lambda_2 = -12.9541$	$v_2 = (0.6880,\ \text{-}0.7257)$
	$C3.1^{***} = (0.3074,\ 0.6847)$	$\lambda_1 = -6.9728$	$v_1 = (0.6368,\ 0.7710)$
		$\lambda_2 = 3.3669$	$v_2 = (0.7025,\ \text{-}0.7117)$
	$D3^{***} = (0.0787,\ 0.5815)$	$\lambda_1 = 3.6467$	$v_1 = (0.0797,\ \text{-}0.9968)$
		$\lambda_2 = -3.19979 * 10^-8$	$v_2 = (\text{-}0.9127,\ 0.4086)$