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Opinion formation and targeting when persuaders have extreme and centrist opinions

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Abstract

We consider a model of competitive opinion formation in which three persuaders characterized by (possibly unequal) persuasion impacts try to influence opinions in a society of individuals embedded in a social network. Two of the persuaders have the extreme and opposite opinions, and the third one has the centrist opinion. Each persuader chooses one individual to target, i.e., he forms a link with the chosen individual in order to spread his own "point of view" in the society and to get the average long run opinion as close as possible to his own opinion. We examine the opinion convergence and consensus reaching in the society. Also the case when the persuaders choose several targets for diffusion of information is discussed. We study the existence and characterization of pure strategy Nash equilibria in the game played by the persuaders with equal impacts. This characterization depends on influenceability and centrality of the targets. We discuss the effect of the centrist persuader on the consensus and symmetric equilibria, compared to the framework with only two persuaders having the extreme opinions. When the persuasion impacts are unequal with one persuader having a sufficiently large impact, the game has only equilibria in mixed strategies.

Keywords : Social network, opinion formation, consensus, targeting, extreme persuader, centrist persuader.
JEL Classification : D85, D72, C72.

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1 Introduction

Social networks play a central role in most of our everyday activities, communicating and exchanging information, sharing knowledge, research and development, advertisement, among many others. A process that can perfectly be modeled by social networks is the one of opinion formation. The opinions result from interactions with other individuals that hold views on given issues. In the seminal model on opinion formation introduced by DeGroot (1974), individuals update their opinions by taking weighted averages of their "neighbours", i.e., people that they are connected to in the network. An accompanying question being particularly important, e.g., in lobbying, political campaigning, marketing, or counter-terrorism, is how to identify optimal targets to achieve social impact. Indeed, the reliance on others to form opinions lies at the heart of advertising (Bimpikis et al. (2016)), efforts to make people aware of different issues, preventing criminal social groups and organizations (Ballester et al. (2006)), or attempts of capturing votes in elections. In economics such models are used to study competition between firms and product differentiation. In political science, they are applied for determining equilibrium outcomes of electoral competitions.

We consider a game of competitive opinion formation in a society played by three competing persuaders that have different opinions on a certain issue. The society consists of individuals having their own opinions on that issue and updating them like in DeGroot (1974), i.e., by taking weighted averages of individuals' opinions that they listen to. The opinion is a real number between 0 and 1, and can be interpreted as the intensity of the opinion "yes". Our point of departure for the present paper is Grabisch et al. (2018) who extend the DeGroot model by introducing two persuaders (called external players in their paper) with the extreme opinions 0 and 1. In the present paper, we introduce a third persuader which has the centrist opinion $\frac{1}{2}$. Each persuader chooses one individual to target. Targeting in this setting means forming a link with that individual in order to make the average opinion in the society as close as possible to the persuader's own opinion. The persuaders are characterized by (possibly unequal) persuasion impacts. The higher the impact of a persuader targeting an individual, the more this individual takes the persuader's opinion into account when updating his own opinion.

The main objective of the present work is to study the effects of entering the additional centrist persuader into competition between the two extremist persuaders. First, we examine the opinion convergence and consensus reaching in the society targeted by the three persuaders. Is it possible to obtain a limit opinion vector? Can the society reach a consensus meaning that every individual has the same opinion? If so, how does such a consensus look like? More specifically, how does the presence of the centrist persuader change the convergence and consensus reaching in the society? In order to consider competition between the three persuaders, we define a noncooperative game played by the persuaders with strategies being target individuals and study the existence and characterization of pure strategy Nash equilibria. Grabisch et al. (2018) obtain a constant sum game where players have opposite interests. Our extended game cannot be considered as a constant sum game anymore, and hence we derive new expressions for the payoffs, appropriate for the extended setting. A number of new questions arises. How can the centrist persuader affect optimal strategies of the extreme persuaders determined in Grabisch et al. (2018)? How do characteristics of the key (i.e., targeted) individuals change when the third persuader enters into the play? Which network structures appear to be consistent with the equilibrium in pure strategies?

The extension of Grabisch et al. (2018) by introducing the third persuader with the centrist opinion has a number of consequences on consensus reaching in the society and Nash equilibria of the noncooperative game played by the persuaders. The presence of the centrist persuader preserves opinion convergence but changes the long run opinions of the society and the consensus. When the three persuaders choose the same target, a consensus exists and is determined by the three persuasion impacts. If the impact of the centrist persuader is vanishing, we recover the consensus with only two extreme persuaders. When the impact of one of the persuaders is much larger than these of the others, the consensus approaches the opinion of the high-impact persuader. Moreover, when all three persuaders target the same individual, the presence of the centrist one improves the situation of the weaker extreme persuader in the sense that consensus moves closer to the opinion of the smaller-impact persuader. Although in the paper we focus on the one-target framework, we also briefly discuss the case when the persuaders can target more than one individual. When the number of targets is the same, the convergence result is preserved, and when additionally the persuaders target the same individuals, the consensus result remains valid, independently of the common number of targets.

By using some notions and definitions given in Grabisch et al. (2018), we characterize equilibria in our three-persuader setting. The two key concepts are centrality (also called influence or intermediacy) and influenceability. An individual is more central than another individual if his influence on others reaches the network before the influence of the another individual. Influenceability of an individual means that he listens less to others, and hence it can be easier to influence him by an additional opinion. We focus our analysis on the case with equal impacts and find that both centrality and relative influenceability are important, and the target individuals are completely characterized by these two notions. More precisely, conditions for the existence of symmetric Nash equilibria of the game played by the three equal-impact persuaders is that the relative influence of a potential target must be at least twice higher than the one of any other individual in the network. Strong-impact persuaders must take into account the presence of the new centrist one. When comparing the results to Grabisch et al. (2018), the persuaders are demanding higher centrality from their potential targets to compensate the impact of the new persuader. However, when the persuaders have weak impact, the conditions for Nash equilibria are the same as for the case with only two extreme persuaders. If the persuasion impacts are unequal and one persuader's impact is sufficiently large, then the game has only equilibria in mixed strategies. Besides symmetric equilibria, we also examine non-symmetric Nash equilibria. In particular, we deliver some necessary conditions for the existence of a non-symmetric equilibrium when the persuaders are equally strong. Moreover, we present many numerical examples that illustrate our results.

The leading assumption of the paper that each persuader targets only one individual covers many real-life situations with one target who is a kind of outstanding and influencing master. It is a well-known practice when a celebrity (actor/actress, sportsman/sportswoman, singer, model, showman, etc.) becomes an ambassador or a "face" of the brand. In this way, companies take into consideration the activity of the potential advertiser, his or her popularity, and the number of followers in social media. A good example is Ambassador Marketing that is a form of "word of mouth" marketing, where a person with specific influence or expertise participates in a brand's marketing strategy, by presenting the brand in a way that encourages the audience to purchase a product. Usually the ambassador leverages his or her own popularity on social media platforms to drive the value.

The introduction of the centrist persuader with a specific position that involves balance, neutrality, and equal combination of the extreme positions makes the theoretical framework richer. It can give a more realistic explanation of the political issues, where ordering election candidates on a line and the presence of a centrist candidate is a quite usual assumption. A good example comes from the last French Presidential Elections with the current President seen as centrist. Also some economic spectrum can be covered by our modeling, where three parties can be seen as three main firms that differ from each other by production, work, and distribution. They can compete over marketing campaigns, product adoption, firm allocations, etc. While the framework of the persuaders with extreme and centrist opinions can find many real-life applications, an extension to many more persuaders does not seem so appealing in reality. There are numerous examples with a small number of persuaders, in particular with three persuaders, e.g., when well-known people use only iOS/Linux/Android software, drive German/British/Japanese cars, wear American/Italian/French brands, etc.

Consider Adidas, Nike and Puma, which are three main football equipment manufacturers. Following our model, we can assume that Adidas and Nike are extreme persuaders since they are in daily battle with each other, and Puma can be seen as the centrist persuader. Each of these brands has representatives from the football world: Adidas has a contract with Lionel Messi, Nike – with Cristiano Ronaldo, and Puma – with Antoine Griezmann. In real-world examples, usually important persuaders do not target the same "face" of the brand. We notice a similar feature in our theoretical investigations, as there is no symmetric equilibrium in a perfectly symmetric network with three equally strong persuaders.

In the paper, we assume that the network is fully observed. While this assumption somehow restrict the model applicability to large size networks, it is quite realistic for smaller networks, where the relationships between individuals are easily observed. For instance, this can be the case in committees and smaller institutions whose members work together for a longer period and establish trust relations that become known to everybody.

Our model can also be applied to mobile operators. In most of the countries, the market of firms that provide mobile services is restricted to three or four large companies. In particular, they can be divided into three categories: company A with excellent coverage and high quality services (it settles high prices compared to competitors, but offers the best connectivity, strongest reliability, and highest average speeds across both urban and rural areas), company B (for average price, it gives unlimited talk, text and data with a very high speed, but company's rural coverage is substandard), and company C (low-cost provider, i.e., it settles low price with weak network coverage and its services are not so good as its competitors' products). Following our model, companies A, B and C can be seen as of three different categories: 1 stands for the best and expensive product, $\frac{1}{2}$ is seen as average and affordable, and 0 reflects non-reliable and cheap services. The example exhibits several specific features relevant to our model. The number of consumers is finite, i.e., restricted by network coverage zone and long-term users. Also, people interacting with each other decide which company to choose. Their decision is mostly driven by the recommendations and opinions of their friends. Moreover, the market is saturated, i.e., almost every person is already a customer of at least one mobile company. Hence, competitors target a small number of potential consumers, and therefore targeting the same agent is a common practice.

The paper is organized as follows. The model is introduced in Section 2. Section 3 concerns the opinion convergence and consensus reaching. In Section 4 we define the noncooperative game played by the persuaders and present the equilibrium analysis. More precisely, we determine the equilibrium conditions for the case when the persuasion impacts are the same and briefly discuss the case of the unequal persuasion impacts. The related literature is surveyed in Section 5. Section 6 presents concluding remarks. The Appendices in Section 7 present proofs of the main results (Appendix 7.1), a discussion of the case when the persuaders target more than one individual (Appendix 7.2), where we briefly discuss the convergence and consensus reaching in such a multi-target extended framework, and an elaboration on targeting different individuals (Appendix 7.3).

2 The framework and preliminaries

The model with three persuaders The society consists of a set $N = \{1, ..., n\}$ of individuals who discuss a certain issue. Each individual $i \in N$ has an initial opinion on the issue, given by a real number $x_i(0) \in [0, 1]$ which can be interpreted as the intensity of i's personal opinion "yes" in time 0. The individuals interact with each other, that is, are embedded in a social network, and consequently update their opinions at discrete time $t \in \mathbb{N}$. The society is observed by three persuaders A, B and C who have the fixed opinions 1, $\frac{1}{2}$ and 0, respectively. Each of them chooses one individual in N to form a link with in order to influence the formation of opinions in the society. The individuals targeted by A, B and C are denoted by s_A, s_B and s_C, respectively. The persuaders are characterized by possibly unequal (positive) persuasion impacts λ, γ and μ, respectively, to adjust influence in the society. When persuader A targets the individual s_A, a share λ of the attention of that individual is redirected to A. The same adjustment of influence holds for s_B and s_C being targeted by B and C, with impacts γ and μ, respectively. Table 1 presents the characteristics of the three persuaders.

Persuader	**Fixed opinion**	**Impact**	**Strategy**
A	1	$\lambda > 0$	s_A
B	$\frac{1}{2}$	$\gamma > 0$	s_B
C	0	$\mu > 0$	s_C

Table 1: Characteristics of the persuaders

It is assumed that in the absence of the persuaders, the individuals would update their opinion by using weighted averages of their neighbours' opinions (DeGroot (1974)), that is, according to the rule:

$$\mathbf{x}_N(t) = W\mathbf{x}_N(t-1) = W^t\mathbf{x}_N(0) \tag{2.1}$$

where $W = [w_{ik}]_{i,k \in N}$ is the interaction or influence matrix being row stochastic, i.e., $\sum_{k=1}^{n} w_{ik} = 1$ for every $i \in N$, w_{ik} denotes the weight or trust that individual i assigns to the current opinion of individual k in forming his own opinion in the next period, and $\mathbf{x}_N(t) = [x_1(t), \cdots, x_n(t)]'$ is the opinion (column) vector at time step t.[1]

A directed graph G on N is associated to the matrix W such that there is an arc (i, k) from i to k meaning that i listens to k if and only if $w_{ik} > 0$. We also refer to individual k as a neighbour of i. A walk from node i to node k is a sequence of nodes $(i_1 = i, i_2, \cdots, i_{j-1}, i_j = k)$ such that $w_{i_m i_{m+1}} > 0$ (i.e., there is an arc (i_m, i_{m+1})) for each $m \in \{1, \cdots, j-1\}$. A cycle around i is a walk from i to i which does not pass through i between the starting and ending nodes.[2] A path is a walk such that neither a node nor an arc appears more than once in the sequence. To be consistent with the DeGroot framework (DeGroot (1974)) we assume that the social network defined by the adjacency matrix W is connected, i.e., for every pair of individuals $i, k \in N$ there exists a path from i to k.

In the presence of the persuaders who choose the targets $\mathbf{s} = (s_A, s_B, s_C)$, the $n \times n$ matrix of influence W is extended to a $(n+3) \times (n+3)$ matrix $M_{\lambda,\gamma,\mu}(\mathbf{s})$ such that:

$$M_{\lambda,\gamma,\mu}(\mathbf{s}) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \mathbf{0} \\ 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ \hline \multicolumn{3}{c|}{\Delta_{\lambda,\gamma,\mu}(\mathbf{s}) E_{\lambda,\gamma,\mu}(\mathbf{s})} & \Delta_{\lambda,\gamma,\mu}(\mathbf{s}) W \end{array}\right] \tag{2.2}$$

which similarly to Grabisch et al. (2018) accounts for two effects:

1. the weight renormalization in the presence of the persuaders, given by the weight renormalization matrix $\Delta_{\lambda,\gamma,\mu}(\mathbf{s})$ which is a diagonal matrix with diagonal elements equal to

$$\frac{d_1}{d_1 + \lambda\delta_{1,s_A} + \gamma\delta_{1,s_B} + \mu\delta_{1,s_C}}, \cdots, \frac{d_n}{d_n + \lambda\delta_{n,s_A} + \gamma\delta_{n,s_B} + \mu\delta_{n,s_C}} \tag{2.3}$$

   with d_i being the number of outgoing links of $i \in N$, $\delta_{i,s_j} = 1$ if $i = s_j$ for all $i \in N$, $s_j \in \{s_A, s_B, s_C\}$ and 0 otherwise;

2. the strategic influence given by the matrix

$$E_{\lambda,\gamma,\mu}(\mathbf{s}) = \left[\frac{\lambda}{d_{s_A}} e_{s_A} \;\; \frac{\gamma}{d_{s_B}} e_{s_B} \;\; \frac{\mu}{d_{s_C}} e_{s_C} \right] \tag{2.4}$$

   where e_i denotes the unit vector with coordinate 1 at i.

In the influence matrix $M_{\lambda,\gamma,\mu}(\mathbf{s})$ the first three rows correspond to the weights of the persuaders A, B and C: since they do not listen to the individuals in the society, they put weight 1 for themselves and 0 otherwise. The next n rows correspond to the new weights of the individuals in N adjusted to the extended framework. The individuals targeted by the persuaders

[1] Transposition of column vectors is denoted by $'$, and therefore $\mathbf{x}'_N(t)$ is a row vector.

[2] This definition of a cycle differs from the usual one, which does not allow repetition of any node between the starting and ending nodes.

redistribute their trust among their neighbours and the targeting persuaders: the weights put for the persuaders depend on the persuaders' impacts and are given by $\Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s})$, while the new weights put for the other individuals are $\Delta_{\lambda,\gamma,\mu}(\mathbf{s})W$ instead of W.

In this paper we consider the behaviour of opinions in the society in the long run. The vector of opinions is extended to $\mathbf{x}(t) = [1\ \frac{1}{2}\ 0\ \mathbf{x}_N(t)]'$ where the first three coordinates correspond to the fixed opinions of the persuaders. The opinion updating rule is now determined by

$$\mathbf{x}(t+1) = M_{\lambda,\gamma,\mu}(\mathbf{s})\mathbf{x}(t) = (M_{\lambda,\gamma,\mu}(\mathbf{s}))^{t+1}\mathbf{x}(0) \tag{2.5}$$

which leads to the evolution law for the opinions of the individuals in N given by

$$\mathbf{x}_N(t+1) = \Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s})\begin{bmatrix}1\\ \frac{1}{2}\\ 0\end{bmatrix} + \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W\mathbf{x}_N(t) \tag{2.6}$$

In the next sections we will provide the consensus and equilibrium analysis in the extended framework. First, let us illustrate the introduced matrices by the following example.

Example 1. We consider a society with five individuals who communicate with each other and put weights (trust) on the opinions of individuals they are listening to. Figure 1 shows the society in terms of a directed graph where, for example, individual 1 listens to individuals 2, 4, and 5, and trusts most to the opinion of individual 2 (since the weights are 3/4, 1/8 and 1/8, respectively). At the same time individuals 2, 3 and 4 are listening to individual 1 and, moreover, individual 3 trusts the opinion of individual 1 as much as that of individual 5 (individual 3 has w_{31}=w_{35}). All the assigned trust weights of individuals are given in the adjacency matrix W.

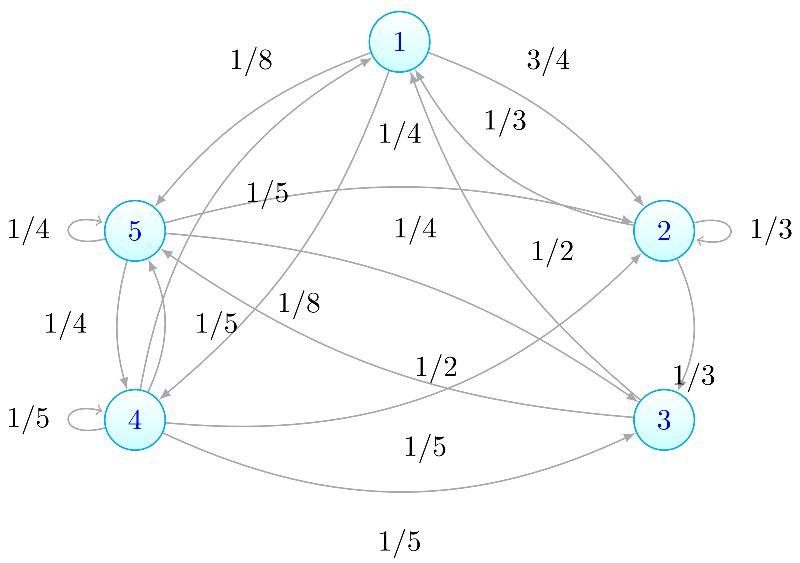

Figure 1: Directed graph representing the society described in Example 1

$$W = \begin{bmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Now assume that there is an intervention with three persuaders who want to influence the opinion formation process in the society. Each of the persuaders chooses one target – one individual in the society.

Situation 1: First, we assume that the three persuaders are equally strong, $\lambda = \gamma = \mu = 1$ and choose the same target. Our goal here is to illustrate the result for the case, where all persuaders decide to choose the same individual. The choice of the target is arbitrary. In the next section, after presenting our results on opinion convergence and consensus reaching, we will elaborate on the fact if this choice matters in the long run. Suppose that all persuaders target individual 2, i.e., $s_A = s_B = s_C = 2$. Then the submatrices $\Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s})$ and $\Delta_{\lambda,\gamma,\mu}(\mathbf{s})W$ are the following:

$$\Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s}) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W = \begin{bmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \tag{2.7}$$

According to (2.2), the $n \times n$ matrix of influence W is extended and becomes $(n+3) \times (n+3)$ matrix $M_{\lambda,\gamma,\mu}(\mathbf{s})$:

$$M_{\lambda,\gamma,\mu}(\mathbf{s}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Comparing the two matrices we can notice that individual 2 redistributes his trust among his neighbours and the targeting persuaders, such that we have the equally assigned weights between them. Other individuals have unchanged weights.

Situation 2: Suppose now that the extreme persuaders target individual 2 while the centrist persuader targets another individual, for instance, individual 4, i.e., $s_A = s_C = 2$ and $s_B = 4$.

Then we obtain:

$$\Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s}) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{5} & 0 & \frac{1}{5} \\ 0 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W = \begin{bmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \tag{2.8}$$

The extended matrix $M_{\lambda,\gamma,\mu}(\mathbf{s})$ can be constructed as in the previous case with the current $\Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s})$ and $\Delta_{\lambda,\gamma,\mu}(\mathbf{s})W$ matrices. As seen in (2.8) individual 2 allocates the weights among five individuals: 3 neighbours and 2 persuaders. At the same time, the weights of individual 4 are distributed by taking into account the central persuader.

Situation 3: Assume now that $s_A = s_C = 2$, $s_B = 4$, but the persuaders have different impacts λ, γ and μ. Take, for instance, $\lambda = 4$, $\gamma = 3$ and $\mu = 8$. We get:

$$\Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s}) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{4}{15} & 0 & \frac{8}{15} \\ 0 & 0 & 0 \\ 0 & \frac{3}{8} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W = \begin{bmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \tag{2.9}$$

This case studies the asymmetric impacts of the persuaders. From the given matrices we can see that the impacts have an essential role in forming the opinion of targeted agents. Consider individual 2 who puts $\frac{4}{15}$ and $\frac{8}{15}$ for his persuaders. It makes sense, because as the impact parameters show persuader C is twice influenceable as persuader A. Note, that the central persuader has increased his impact as well, and therefore the trust he was given becomes almost 2.5 times higher than before.

When interpreting this 5-person society in terms of our mobile operators example given in the introduction, we deal with 5 individuals who are discussing which mobile operator to join. They take differently into account their friends' opinions, i.e., the trust weights are not equal. The higher the weight the more likely the individual adopts the same operator as his friend. In Situation 1, the attention of the target to the companies is divided equally, as they target the same individual and influence him in the same manner. In Situation 2, companies A and C decide for the same target while company B is targeting individual 4. Situation 3 is different from the previous case and shows what happens when companies put different levels of effort to attract the customers. Company B has the lowest impact, but by targeting alone an individual, it gets higher trust weight than companies A and C that share the attention of individual 2.

Influenceability and intermediacy In this paper we consider an extension of the model of strategic influence with two external players having extreme opinions (Grabisch et al. (2018)) to a framework with three persuaders, by adding a persuader with the centrist opinion. We recall some other crucial concepts used in the initial model (Grabisch et al. (2018)) that will be related to the characterization of the targets. For any walk $p = (i_1, ..., i_m)$ in G, we denote by $w(p)$ its "weight" measured according to W, i.e.,

$$w(p) := \prod_{j=1}^{m-1} w_{i_j, i_{j+1}} \tag{2.10}$$

Moreover, for any two individuals i, k in the society N, let C_i^k denote the set of cycles around i that pass through k, and B_i^k the set of walks that start from any node $\neq i$, end up in i, and go through k. Let

$$c_i^k := \sum_{p \in C_i^k} w(p) \qquad\qquad b_i^k := \sum_{p \in B_i^k} w(p) \tag{2.11}$$

The quantity c_i^k accounts for the self-feedback (echo) that individual i receives of his opinion through the network. The larger c_i^k is, the more individual k interferes with this self-reinforcement process and hence, the lesser is the influence that can be exerted on i by a persuader. Furthermore, the quantity $d_i c_i^k$ measures the *influenceability of individual* i, given that k is targeted by another persuader, where d_i is i's out-degree, i.e., the number of individuals that i listens to. This is a decreasing measure. The larger d_i is, the more opinions individual i takes into account and the lesser/slower he can be influenced by an additional opinion. Hence, the lower $d_i c_i^k$ is, the more influenceable i is, i.e., the higher his influenceability is.

The quantity b_i^k accounts for the *centrality (influence, intermediacy) of* k *relatively to* i, i.e., it measures the extent to which k can interpose himself between i and other individuals, i.e., the extent to which the influence of individual k reaches the network before this of i.

c_i^k and b_i^k have some probabilistic interpretations. If the influence travels across the network according to the probabilities given by W, then c_i^k is the probability for i to be reached by the influence of k before he receives the self-feedback of his own opinion. Accordingly, b_i^k is the sum of the probabilities for the $n-1$ individuals other than i to be reached by the influence of k before this of i.

3 Convergence of opinions and consensus reaching

Our first result concerns the convergence of opinions in the influence model with three persuaders. When the society gets a new persuader, the one with the centrist position, the opinion convergence is preserved in the society, i.e., opinions of the individuals do converge in long run. However, the long run opinions are obviously different from the ones reached in a society with only two persuaders having the extreme positions. More precisely, the following proposition holds:

Proposition 1. *For any initial vector of opinions* $\boldsymbol{x}(0) := [1 \ \frac{1}{2} \ 0 \ \boldsymbol{x}_N(0)]'$, *we have*

$$\lim_{t \to +\infty} (M_{\lambda,\gamma,\mu}(\boldsymbol{s}))^t \begin{bmatrix} 1 & \frac{1}{2} & 0 & \boldsymbol{x}_N(0) \end{bmatrix}' = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \overline{\boldsymbol{x}}_N(\boldsymbol{s}) \end{bmatrix}' \tag{3.1}$$

where

$$\overline{\boldsymbol{x}}_N(\boldsymbol{s}) = [I - \Delta_{\lambda,\gamma,\mu}(\boldsymbol{s}) W]^{-1} \Delta_{\lambda,\gamma,\mu}(\boldsymbol{s}) \left(\frac{\lambda}{d_{s_A}} e_{s_A} + \frac{\gamma}{2 d_{s_B}} e_{s_B} \right) \tag{3.2}$$

In the model with two persuaders having the opinions 1 and 0, and the impacts λ and μ, respectively, Grabisch et al. (2018) prove the convergence result with

$$\overline{\mathbf{x}}_N(\mathbf{s}) = [I - \Delta_{\lambda,\mu}(\mathbf{s})W]^{-1} \frac{\lambda}{d_{s_A}} \Delta_{\lambda,\mu}(\mathbf{s}) e_{s_A}$$

The presence of the centrist persuader leads to different $\overline{\mathbf{x}}_N(\mathbf{s})$ compared to the corresponding results in Grabisch et al. (2018). The coefficient $\frac{1}{2}$ comes from the vector of opinions where the first three coordinates are fixed points of the persuaders. The additional component is related to the presence of the centrist persuader with the impact γ, and $\frac{1}{2}$ indicates his "ideal" opinion. Similarly to Grabisch et al. (2018), in our extended framework with three persuaders the asymptotic opinions of the individuals are independent of their vector of initial opinions. They are determined by the respective targets of the three persuaders, since $\overline{\mathbf{x}}_N(\mathbf{s}) \in [0, 1]^n$ determined by (3.2) depends on the whole vector $\mathbf{s}$ and the persuasion impacts λ, γ, and μ.

The next issue concerns the effect of the centrist persuader on reaching a consensus among the society members. In other words, can all individuals end up with the same opinion in long run, and if so, how does their opinion look like? It appears that if the three persuaders choose the same target, then the long run opinion in the society converges towards a consensus $\alpha \in [0, 1]$ among the individuals. The consensus is determined by the three persuasion impacts.

Proposition 2. *If $s_A = s_B = s_C$, then the individuals in N reach a consensus α given by*

$$\alpha = \frac{2\lambda + \gamma}{2(\lambda + \gamma + \mu)} \tag{3.3}$$

In particular, if $\lambda = \mu$, then the consensus is $\alpha = \frac{1}{2}$.

We can draw a number of intuitive conclusions from Proposition 2. In the case when all three persuaders target the same individual, the society reaches a consensus which depends on the persuaders' impacts. In particular, if the extreme persuaders have the equal impact $\lambda = \mu$, then the individual who receives three "types" of information from each of the persuaders, takes equally into account the opinions 0 and 1 of the extreme persuaders. At the end, the consensus of $\frac{1}{2}$ occurs in the society, independently of the impact of the centrist persuader B. Moreover, note that $\lambda \geq \mu$ is also the condition for $\alpha \geq \frac{1}{2}$, since in case of the same target we have $\lambda \geq \mu$ if and only if $\alpha \geq \frac{1}{2}$. Hence, when all persuaders target the same individual and the extreme ones have unequal impacts, then the consensus is closer to the opinion of the stronger extreme persuader.

In the model Grabisch et al. (2018) with two extreme persuaders targeting the same individual, the society reaches a consensus given by $\alpha = \frac{\lambda}{\lambda+\mu}$. We recover this result from (3.3) when the centrist persuader in the extended model has the vanishing impact $\gamma \to 0$. On the contrary, if the centrist persuader is much stronger than the two extreme ones, i.e., if $\gamma \to +\infty$ and $\lambda, \mu \in \mathbb{R}_+$, then the consensus is equal to $\frac{1}{2}$, the opinion of the centrist persuader. Similarly, when $\lambda \to +\infty$ and $\gamma, \mu \in \mathbb{R}_+$, the consensus is equal to 1 (A's opinion), while under $\mu \to +\infty$ and $\lambda, \gamma \in \mathbb{R}_+$, the consensus approaches 0 (C's opinion). Furthermore, note that

$$\frac{2\lambda + \gamma}{2(\lambda + \gamma + \mu)} > \frac{\lambda}{\lambda + \mu} \quad \text{if and only if} \quad \lambda < \mu$$

This means that when all three persuaders target the same individual, independently of the impact of the centrist persuader, his presence in the society always improves the situation of the weaker extreme persuader w.r.t. the situation when the centrist persuader is absent. More precisely, the appearance of the centrist persuader has a "balancing" effect and moves the consensus opinion closer to the ideal point of the persuader with the smaller impact.

When persuaders A and C target the same individual, then the society ends up in a consensus, even if the centrist persuader targets another individual and independently of his own impact, but only if the extreme persuaders are equally strong. In this case, the consensus is equal to $\frac{1}{2}$. More precisely, the following holds true.

Proposition 3. *If $s_A = s_C$ and $\lambda = \mu$ then the individuals in N reach a consensus $\alpha = \frac{1}{2}$.*

The individual targeted by persuaders A and C listens to both of them. He recounts his trust weights, and since impacts are equal ($\lambda = \mu$), spreads the opinion of $\frac{1}{2}$. At the same time, the individual targeted by persuader B shares the same opinion. Consequently, the society reaches the consensus $\frac{1}{2}$, similarly as in the absence of the centrist persuader. In other words, if the extreme persuaders are equally important and target the same individual, then the appearance of the centrist persuader obviously does not change the consensus. We present an illustrative example.

Example 1 (continued)**.** Let us examine the convergence and consensus reaching in the society introduced in Example 1.

Situation 1: Let $s_A = s_B = s_C = 2$ and $\lambda = \gamma = \mu = 1$. The vector $\overline{\mathbf{x}}_N(\mathbf{s})$ is obtained from (2.6), letting $\mathbf{x}_N(t+1) = \mathbf{x}_N(t) = \overline{\mathbf{x}}_N(\mathbf{s})$. The solution of

$$\overline{\mathbf{x}}_N(\mathbf{s}) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \frac{1}{2} 0 \end{bmatrix}' + \begin{bmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \overline{\mathbf{x}}_N(\mathbf{s})$$

is $\overline{x}_i(\mathbf{s}) = \frac{1}{2}$ for $i \in \{1, 2, 3, 4, 5\}$, i.e., the society converges to a consensus $\alpha = \frac{1}{2}$. Obviously, the solution is consistent with Proposition 2. Note that, while the opinions in the long run do depend on the specific targets as shown by Proposition 1, by virtue of Proposition 2 the society would reach the same consensus $\alpha = \frac{1}{2}$ independently of the target, if all persuaders are equally strong and choose the same individual.

Situation 2: In case when $s_A = s_C = 2$, $s_B = 4$, we obtain the solution of

$$\overline{\mathbf{x}}_N(\mathbf{s}) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{5} & 0 & \frac{1}{5} \\ 0 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \frac{1}{2} 0 \end{bmatrix}' + \begin{bmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \overline{\mathbf{x}}_N(\mathbf{s})$$

equal to $\overline{x}_i(\mathbf{s}) = \frac{1}{2}$ for $i \in \{1, 2, 3, 4, 5\}$. The payoffs of the persuaders are the same as in the previous case. Consistently with Proposition 3, since $\lambda = \mu$, the society reaches the consensus equal to $\alpha = \frac{1}{2}$, despite the fact that the impact of the centrist persuader is different from the one of the extreme persuaders. Similarly as in the previous situation, the same consensus would be reached, if persuaders A and C remain having the same impact and target another (but common) individual.

Situation 3: We have $s_A = s_C = 2$, $s_B = 4$, but $\lambda = 4$, $\gamma = 3$ and $\mu = 8$. The solution of

$$\overline{\mathbf{x}}_N(\mathbf{s}) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{4}{15} & 0 & \frac{8}{15} \\ 0 & 0 & 0 \\ 0 & \frac{3}{8} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \frac{1}{2} 0 \end{bmatrix}' + \begin{bmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \overline{\mathbf{x}}_N(\mathbf{s})$$

is equal to $\overline{x}_1(\mathbf{s}) = 0.3511$, $\overline{x}_2(\mathbf{s}) = 0.3366$, $\overline{x}_3(\mathbf{s}) = 0.3614$, $\overline{x}_4(\mathbf{s}) = 0.4173$, and $\overline{x}_5(\mathbf{s}) = 0.3718$. Since the impact μ of the third persuader is twice of the impact λ of the first one, the long run average opinion is biased toward the first half of opinion domain. In other words, when $\lambda \neq \mu$, we get the long run opinions convergence, but there is no consensus $\overline{x}_i(\mathbf{s}) \neq \overline{x}_k(\mathbf{s})$ for some $i, k \in N$. In this situation, the choice of the specific target does matter in the long run. In other words, while opinions of the individuals would always converge in the long run as insured by Proposition 1, targeting another individual by unequally important persuaders could lead to different long run opinions.

We come back to our example with mobile companies A, B and C. In situation 1 targeting the same individual and putting the same level of effort is suboptimal for the high-quality and low-cost companies, since in the long run individuals decide for a product from the average company B. In situation 2, as shown in Proposition 3, even if the extreme companies A and C target independently from company B, they will still lose consumers as long as both of them keep the equal impact. In the last situation where the high-quality company A, the average company B and the low-cost operator C have the impacts 4, 3, and 8, respectively, individuals do not unanimously agree to choose company B, but are heterogeneous in their opinions. They are more likely to adopt products of companies B and C.

4 Nash equilibrium of the model

Payoffs and the aggregate opinion We consider a game $\mathcal{G}_{\lambda,\gamma,\mu}$ played by the three persuaders, with their set of strategies being N, i.e., the strategies of A, B and C are the targeted individuals s_A, s_B and s_C, respectively. Each persuader aims at bringing the asymptotic average opinion in the society as close as possible to his own opinion (1 for persuader A, $\frac{1}{2}$ for persuader B and 0 for persuader C), i.e., at minimizing the distance between the asymptotic average opinion in the society and his own "ideal" point. In other words, our game-theoretic model of competition between the persuaders is a system of minimization problems, where the persuaders' goal is to minimize their payoffs, given a strategy profile $\mathbf{s} = (s_A, s_B, s_C) \in N \times N \times N$ defined in the following way:

$$
\begin{aligned}
\pi^A_{\lambda,\gamma,\mu}(s_A, s_B, s_C) &= \left(1 - \frac{1}{n}\mathbf{1}'\overline{\mathbf{x}}_N(\mathbf{s})\right)^2 \\
\pi^B_{\lambda,\gamma,\mu}(s_A, s_B, s_C) &= \left(\frac{1}{2} - \frac{1}{n}\mathbf{1}'\overline{\mathbf{x}}_N(\mathbf{s})\right)^2 \\
\pi^C_{\lambda,\gamma,\mu}(s_A, s_B, s_C) &= \left(\frac{1}{n}\mathbf{1}'\overline{\mathbf{x}}_N(\mathbf{s})\right)^2
\end{aligned}
\tag{4.1}
$$

where $\overline{\mathbf{x}}_N(\mathbf{s})$ is given by (3.2). For convenience, we introduce the notation

$$\tilde{x}_N(\mathbf{s}) := \mathbf{1}'\overline{\mathbf{x}}_N(\mathbf{s}) = \sum_{i \in N} \overline{x}_i(\mathbf{s})$$

for the aggregate opinion formed in the society.[3] The following results determine $\tilde{x}_N(\mathbf{s})$, i.e., equivalently, the persuaders' payoffs for some strategy profiles in terms of the persuaders' impacts, the individuals' centrality and influenceability recalled in Section 2.

Theorem 1. *The payoffs of persuaders A, B and C, given the strategy profile* $\mathbf{s} = (s_A, s_B, s_C)$ *are as follows:*

(i) If $s_A = s_B = s_C = i$*, i.e., if all three persuaders target the same individual* i*, then:*

$$\tilde{x}_N(i,i,i) = \frac{n(2\lambda + \gamma)}{2(\lambda + \gamma + \mu)} \tag{4.2}$$

(ii) If $s_A = s_C = i$ *and* $s_B = k \neq i$*, i.e., if the two extreme persuaders target the same individual* i *and the centrist one targets a different individual* k*, then:*

$$\tilde{x}_N(i,k,i) = \frac{2\lambda(\gamma b^i_k + d_k c^i_k n) + \gamma((\lambda + \mu)b^k_i + d_i c^k_i n)}{2(\gamma d_i c^k_i + (\lambda + \mu)(d_k c^i_k + \gamma))} \tag{4.3}$$

(iii) If $s_A = k$ *and* $s_B = s_C = i \neq k$*, i.e., if the persuader with the opinion 1 targets an individual* k*, while the remaining persuaders target the same individual* i *but different from* k*, then:*

$$\tilde{x}_N(k,i,i) = \frac{2\lambda((\gamma + \mu)b^k_i + d_i c^k_i n)) + \gamma(\lambda b^i_k + d_k c^i_k n)}{2(\lambda d_i c^k_i + (\gamma + \mu)(d_k c^i_k + \lambda))} \tag{4.4}$$

(iv) If $s_A = s_B = i$ *and* $s_C = k \neq i$*, i.e., if the first two persuaders target* i *and the one with the opinion 0 targets a different individual* k*, then:*

$$\tilde{x}_N(i,i,k) = \frac{(2\lambda + \gamma)(\mu b^i_k + d_k c^i_k n)}{2(\mu d_i c^k_i + (\lambda + \gamma)(d_k c^i_k + \mu))} \tag{4.5}$$

[3] Note that the payoff function of each persuader rewards the average opinion that is close to the persuader's own opinion without taking into account the distribution of opinions. One could also consider the payoffs that depend on the distance to each individual's opinion, instead of their aggregate. Note that we have $\pi^A_{\lambda,\gamma,\mu}(s_A, s_B, s_C) = \frac{1}{n^2}\left(\sum_{i \in N}(1 - \overline{x}_i(\mathbf{s}))\right)^2$, $\pi^B_{\lambda,\gamma,\mu}(s_A, s_B, s_C) = \frac{1}{n^2}\left(\sum_{i \in N}(\frac{1}{2} - \overline{x}_i(\mathbf{s}))\right)^2$, and $\pi^C_{\lambda,\gamma,\mu}(s_A, s_B, s_C) = \frac{1}{n^2}\left(\sum_{i \in N}\overline{x}_i(\mathbf{s})\right)^2$.

The first result (i) of Theorem 1 is consistent with Proposition 2. If the three persuaders target the same individual i, then the aggregate opinion in the society depends fully on the persuaders' impacts and is equal to $\tilde{x}_N(i,i,i) = \frac{n(2\lambda+\gamma)}{2(\lambda+\gamma+\mu)}$, where $\frac{2\lambda+\gamma}{2(\lambda+\gamma+\mu)}$ is the consensus reached in the society. In this case, the payoffs are equal to:

$$\begin{aligned} \pi_\lambda^A(i,i,i) &= \left(\frac{2\mu+\gamma}{2(\lambda+\gamma+\mu)}\right)^2 \\ \pi_\lambda^B(i,i,i) &= \left(\frac{\mu-\lambda}{2(\lambda+\gamma+\mu)}\right)^2 \\ \pi_\lambda^C(i,i,i) &= \left(\frac{2\lambda+\gamma}{2(\lambda+\gamma+\mu)}\right)^2 \end{aligned} \tag{4.6}$$

In cases (ii), (iii) and (iv) of Theorem 1, i.e., when two persuaders target the same individual and the remaining one chooses another individual, the aggregate opinion depends on the persuaders' impacts as well as on the intermediacy and influenceability of the targets. Moreover, the second result (ii) of Theorem 1 is consistent with Proposition 3. If $s_A = s_C = i$, $s_B = k \neq i$, and $\lambda = \mu$, then the society reaches the consensus $\frac{1}{2}$, independently of the centrist persuader's impact. Applying $\lambda = \mu$ to (4.3) gives the aggregate opinion $\tilde{x}_N(i,k,i) = \frac{n}{2}$.

While the concepts of intermediacy and influenceability, together with the persuasion impacts determine the aggregate opinion when at least two persuaders target the same individual, these concepts are not sufficient for determining the aggregate opinion $\tilde{x}_N(i,j,k)$ with i, j and k being all different. The case when the persuaders target three different individuals is briefly mentioned in the Appendix 7.3. Later in this section we deliver necessary conditions for a non-symmetric Nash equilibrium for the case of equal persuasion impacts. We also present numerous examples.

To get more insights from the results of Theorem 1, let us discuss some properties of our model with respect to the key parameters: the impact of persuaders, and the influence and influenceability of individuals in the network.

Fact 1. *Suppose that three persuaders are targeting the same individual, i.e., $s_A = s_B = s_C = i$ for some $i \in N$. The marginal effects of a change of the impact λ, γ, μ on the aggregate opinion are the following:*

$$\frac{\partial \tilde{x}_N(i,i,i)}{\partial \lambda} > 0, \quad \frac{\partial \tilde{x}_N(i,i,i)}{\partial \mu} < 0, \quad \textit{and moreover} \quad \frac{\partial \tilde{x}_N(i,i,i)}{\partial \gamma} \gtreqless 0 \quad \Leftrightarrow \quad \mu \gtreqless \lambda \tag{4.7}$$

From (4.2) we can conclude that, by increasing his impact λ, persuader A gets more attention from individual i and therefore increases the aggregate opinion. While persuader A pulls the aggregate opinion to his own opinion that equals 1, persuader C wants the aggregate opinion to be as low as possible such that it reaches his ideal opinion 0. These observations confirm the first two inequalities in (4.7). Also the remaining result in (4.7) has its natural interpretation. When the impacts of the extreme persuaders are equal, by increasing or decreasing his own impact the centrist persuader has no effect on the aggregate opinion. However, when persuader A is stronger than persuader C, the centrist persuader lowers the aggregate opinion by increasing γ. On the contrary, B increases $\tilde{x}_N(i,i,i)$ by increasing his impact when persuader C is stronger than A. Hence, the centrist persuader can be seen as a balancing "player", not allowing the aggregate opinion move far away from the center.

Fact 2. *The marginal effects of a change of the intermediacy of one individual relatively to another individual on the aggregate opinion are the following:*

$$\begin{aligned}&\textit{Let}\ \ s_A = s_C = i \ \ \textit{and} \ \ s_B = k \neq i.\\ &\quad \textit{Then} \ \ \frac{\partial \tilde{x}_N(i,k,i)}{\partial b_k^i} \gtreqless 0 \ \ \Leftrightarrow \ \ \lambda \gtreqless \mu, \ \ \textit{and} \ \ \frac{\partial \tilde{x}_N(i,k,i)}{\partial b_i^k} \gtreqless 0 \ \ \Leftrightarrow \ \ \mu \gtreqless \lambda\end{aligned} \tag{4.8}$$

$$\begin{aligned}&\textit{Let}\ \ s_A = k \ \ \textit{and} \ \ s_B = s_C = i \neq k.\\ &\quad \textit{Then} \ \ \frac{\partial \tilde{x}_N(k,i,i)}{\partial b_k^i} \gtreqless 0 \ \ \Leftrightarrow \ \ \gamma \gtreqless 2\mu, \ \ \textit{and} \ \ \frac{\partial \tilde{x}_N(k,i,i)}{\partial b_i^k} \gtreqless 0 \ \ \Leftrightarrow \ \ 2\mu \gtreqless \gamma\end{aligned} \tag{4.9}$$

$$\begin{aligned}&\textit{Let}\ \ s_A = s_B = i \ \ \textit{and} \ \ s_C = k \neq i.\\ &\quad \textit{Then} \ \ \frac{\partial \tilde{x}_N(i,i,k)}{\partial b_k^i} > 0 \ \ \textit{and} \ \ \frac{\partial \tilde{x}_N(i,i,k)}{\partial b_i^k} < 0\end{aligned} \tag{4.10}$$

(4.8), (4.9) and (4.10) concern the effect of the influence (intermediacy) on the aggregate opinion. Note that when two persuaders target the same individual i and the third one chooses another individual k, the relative intermediacies of the two targets have an opposite effect on the aggregate opinion in the society. More precisely, if the extreme persuaders have a common target i, then this effect depends on the relation between the impacts of the extreme persuaders. When A is stronger than C, an increase of the intermediacy of i relatively to k (of k relatively to i, respectively) increases (decreases, respectively) the aggregate opinion. The effect is opposite when C is stronger than A, as stated in (4.8). The case (4.9) with B and C targeting the same individual i and A choosing another individual k is analogous, i.e., the effects of a change of the intermediacies depend on the impacts of B and C, except that the persuasion impact of B is now compared to the double impact of C. Finally, as presented in (4.10), when A and B target the same individual i and C targets another individual k, the aggregate opinion is increasing w.r.t. the intermediacy of i relatively to k, and decreasing w.r.t. to the influence of k relatively to i.

Fact 3. *Suppose that three persuaders have the same impact, i.e.,* $\lambda = \gamma = \mu$. *The marginal effects of a change of the individual's influenceability on the aggregate opinion are the following:*

$$\textit{If} \ \ s_A = s_C = i \ \ \textit{and} \ \ s_B = k \neq i, \ \ \textit{then} \ \ \frac{\partial \tilde{x}_N(i,k,i)}{\partial (d_i c_i^k)} = \frac{\partial \tilde{x}_N(i,k,i)}{\partial (d_k c_k^i)} = 0 \tag{4.11}$$

$$\textit{If} \ \ s_A = k \ \ \textit{and} \ \ s_B = s_C = i \neq k, \ \ \textit{then} \ \ \frac{\partial \tilde{x}_N(k,i,i)}{\partial (d_i c_i^k)} > 0 \ \ \textit{and} \ \ \frac{\partial \tilde{x}_N(k,i,i)}{\partial (d_k c_k^i)} < 0 \tag{4.12}$$

$$\textit{If} \ \ s_A = s_B = i \ \ \textit{and} \ \ s_C = k \neq i, \ \ \textit{then} \ \ \frac{\partial \tilde{x}_N(i,i,k)}{\partial (d_i c_i^k)} < 0 \ \ \textit{and} \ \ \frac{\partial \tilde{x}_N(i,i,k)}{\partial (d_k c_k^i)} > 0 \tag{4.13}$$

Fact 3 shows how the aggregate opinion changes with an increase/decrease of individuals' influenceability. As stated in (4.11), when the extreme persuaders target the same i and the centrist persuader targets a distinct individual k, neither changing the influenceability of i nor of k has an effect on the aggregate opinion. Here, the equal impact of the persuaders plays a crucial role, since it balances the influence of the extreme persuaders and makes the centrist one indifferent between increasing and decreasing the influenceability of his own target. An increase of $d_i c_i^k$, where i is targeted by persuaders B and C, makes i less attentive to his persuaders. As a consequence, their overall influence in the network goes down and the aggregate influence rises in favor of persuader A. As stated in (4.12), the effect of an increase of $d_k c_k^i$ on the aggregate opinion is the opposite, since the target k of persuader A pays less attention to his persuader with the opinion 1. (4.13) can be interpreted by using a similar argument for the situation with i being targeted by A and B, and k being the target of persuader C with the opinion 0.

Equal persuasion impacts We focus our analysis on pure strategy Nash equilibria in the case when all three persuaders have the same impact, i.e., $\lambda = \gamma = \mu$. We replace $\mathcal{G}_{\lambda,\gamma,\mu}$ by the simplified notation $\mathcal{G}_\lambda$ for the game, and $\pi^A_{\lambda,\gamma,\mu}$, $\pi^B_{\lambda,\gamma,\mu}$, $\pi^C_{\lambda,\gamma,\mu}$ by π^A_λ, π^B_λ, π^C_λ for the payoffs. From equations (4.1), (4.2) and (4.3) we get direct conclusions of Theorem 1. Indeed, if $\lambda = \mu$, then one has for all $i, k \in N$:

$$\pi^A_\lambda(i,i,i) = \frac{1}{4}, \quad \pi^B_\lambda(i,i,i) = 0 \quad \pi^C_\lambda(i,i,i) = \frac{1}{4} \tag{4.14}$$

$$\pi^A_\lambda(i,k,i) = \frac{1}{4}, \quad \pi^B_\lambda(i,k,i) = 0, \quad \pi^3_\lambda(i,k,i) = \frac{1}{4} \tag{4.15}$$

This means that the centrist persuader is indifferent between targeting individual i or individual k, because in both cases the outcome is the same, and the average opinion is equal to his "ideal" opinion. Note that it is true only when the two extreme persuaders have the equal impacts $\lambda = \mu$.

First, we focus our analysis on symmetric Nash equilibria. The following result provides necessary and sufficient conditions for (i,i,i) to be an equilibrium.

Theorem 2. *A profile of strategies (i,i,i) is an equilibrium of the game $\mathcal{G}_\lambda$ if and only if for all $k \in N \setminus \{i\}$*

$$b_k^i - 2b_i^k \geq \frac{n}{\lambda}\left(d_i c_i^k - d_k c_k^i\right) \tag{4.16}$$

The equilibrium condition depends both on the centrality and influenceability of the target i relative to any other individual: (i,i,i) is an equilibrium if for all $k \neq i$, the difference between the influence (intermediacy) of i over k and the double influence of k over i is not smaller than the difference between the influenceability of i and the influenceability of k, scaled by the factor $\frac{n}{\lambda}$. In the model with only two extreme persuaders, Grabisch et al. (2018) get a similar condition for (i,i) to be an equilibrium, but with the expression $\left(b_k^i - b_i^k\right)$ on the left hand side of the inequality. In the extended three-persuader model, the condition to reach the equilibrium (i,i,i) requires more from the intermediacy of i over k than in the framework with only two extreme persuaders (Grabisch et al. (2018)): i must be even more influential (central) among other individuals to compensate impact of two other persuaders.

Condition (4.16) also shows what happens under different multiplier $\frac{n}{\lambda}$. As the number of individuals in the society increases, the relative importance of intermediacy compared to influenceability goes down. Conversely, the relative importance of intermediacy goes up with the level of λ, the impact of the persuaders.

When the persuaders have the same impact, pure Nash equilibria can exist in types of networks that are structurally very different. Common feature of such networks is the presence of an individual or a group of individuals with either high intermediacy or high influenceability. For the game with three competitive persuaders there exist networks with symmetric Nash equilibria in pure strategies (e.g., star networks) and also networks where no symmetric equilibria in pure strategies can be found (e.g., symmetric and circular networks). We show it in the following examples.

Example 2. Consider a perfectly symmetric society, i.e., a network structure such that for all distinct $i, k \in N$, $d_i = d_k$, $c_i^k = c_k^i$, and $b_i^k = b_k^i$. While (i, i) was always an equilibrium in the model with only two extreme persuaders, condition (4.16) does not hold in the extended framework, so that (i, i, i) is not an equilibrium of the game $\mathcal{G}_\lambda$ in perfectly symmetric networks. It is not worth targeting i and sharing the attention of the individual with two other persuaders, since there are other individuals with the same characteristics whose targeting can lead to a better payoff.

Example 3. Condition (4.16) means that a network has to contain a very "powerful" individual in order to get a symmetric equilibrium. We consider a star society, where one central individual is connected to any other individual in the network, i.e., the structure given by $d_i = n-1$ and $d_k = 1$, with individual i being central and all individuals $k \neq i$ being peripheral. We have $c_k^i = 1$, $c_i^k = \frac{1}{n-1}$, $b_k^i = n-1$, $b_i^k = 1$. Hence, (4.16) is always satisfied in such star networks (unless the number of individuals in the society is less than 3).

Example 4. Consider a society interacted in a directed circle, where every individual listens to the next one, and only to him. We have $d_i = 1$ for every $i \in N$. Moreover, for any $k \neq i$, $c_i^k = c_k^i = 1$, $b_k^i = l(k, i)$ and $b_i^k = l(i, k)$, where $l(k, i)$ and $l(i, k)$ are the lengths of the (unique) shortest walk from k to i, and from i to k, respectively. If $\lambda = \gamma = \mu$ then no symmetric equilibrium in pure strategies can exist in such a circular network, similarly to the case with only two extreme persuaders.

Example 5. Consider a society organized in a line network. Such a structure has two types of nodes: $d_1 = d_n = 1$ and $d_j = 2$ for each $j \neq 1, n$. Note that for each $i \in N$, $d_i = d_{n+1-i}$, $c_i^{n+1-i} = c_{n+1-i}^i$ and $b_i^{n+1-i} = b_{n+1-i}^i$. If n is even, then (i, i, i) cannot be a Nash equilibrium, because condition (4.16) is not satisfied for $k = n+1-i$. Consider n odd. We can see from Example 3 that there exists a symmetric equilibrium for $n = 3$. Let $n \geq 5$. Note that by using the same argument as for n even, (i, i, i) cannot be a Nash equilibrium for $i \neq \frac{n+1}{2}$. Consider now $i = \frac{n+1}{2}$ and $k = \frac{n-1}{2}$. We can show that $b_i^k = \frac{n-1}{2}$, $b_k^i = \frac{n+1}{2}$, and therefore $b_k^i - 2b_i^k = \frac{3-n}{2} \leq 0$ for $n \geq 3$. Moreover, $c_i^k = c_k^i = \frac{1}{2}$. Hence, condition (4.16) is not satisfied, and therefore (i, i, i) cannot be a Nash equilibrium for $i = \frac{n+1}{2}$.

Let us consider the two polar cases where the impact of the persuaders is either infinitely large or infinitely small with respect to the normalized influence within the network. We get the following result.

Proposition 4. *(i) For distinct $i, k \in N$:*

$$\lim_{\lambda \to 0} \pi_\lambda^A(k,i,i) = \lim_{\lambda \to 0} \pi_\lambda^C(i,i,k) = \left(\frac{3d_k c_k^i}{2(d_i c_i^k + 2d_k c_k^i)} \right)^2$$

$$\lim_{\lambda \to 0} \pi_\lambda^B(i,k,i) = 0$$

so that (i,i,i) is an equilibrium of the game $\mathcal{G}_\lambda$ as $\lambda \to 0$ if and only if for all $k \in N$

$$d_k c_k^i \geq d_i c_i^k \tag{4.17}$$

(ii) For distinct $i, k \in N$:

$$\lim_{\lambda \to +\infty} \pi_\lambda^A(k,i,i) = \lim_{\lambda \to +\infty} \pi_\lambda^C(i,i,k) = \left(\frac{3b_k^i}{4n} \right)^2$$

$$\lim_{\lambda \to +\infty} \pi_\lambda^B(i,k,i) = 0$$

so that (i,i,i) is an equilibrium of the game $\mathcal{G}_\lambda$ as $\lambda \to +\infty$ if and only if for all $k \in N$

$$b_k^i \geq 2b_i^k \tag{4.18}$$

The first part of Proposition 4 is interpreted in terms of influenceability, and the second part – in terms of centrality. Hence, Proposition 4 says the following:

(i) The strategy profile (i,i,i) is a Nash equilibrium of the game $\mathcal{G}_\lambda$ for a vanishingly small level of impact λ if and only if i is at least as influenceable as any other individual $k \in N$. When the persuaders are of the weak impact, they should rather target highly influenceable individuals, i.e., i with the lower $d_i c_i^k$. Such i does not listen to many other individuals and it is easier and quicker to convince him to follow a new opinion.

(ii) The strategy profile (i,i,i) is a Nash equilibrium of the game $\mathcal{G}_\lambda$ for an arbitrarily large level of impact λ if and only if the relative influence of i is not smaller than the double relative influence of any other individual k. When the level of impact increases, the persuaders should target highly central individuals.

When comparing Proposition 4 to the corresponding result in the presence of only two extreme persuaders (Grabisch et al. (2018)), the optimal behavior of weak-impact persuaders are the same even with a growing number of individuals. However, strong-impact persuaders should take into account the presence of the new centrist persuader, since we have the condition $b_k^i \geq 2b_i^k$ instead of $b_k^i \geq b_i^k$ (condition in the case of only two extreme persuaders).

Next, let us consider the case of non-symmetric equilibria. In Proposition 5 we deliver necessary conditions for a non-symmetric Nash equilibrium (i,j,k) with $j \notin \{i,k\}$ when the persuaders are equally strong. More precisely, we determine a condition for the aggregate opinion $\tilde{x}_N(i,j,k)$ and conditions for the network structure that can admit a non-symmetric equilibrium, expressed in terms of the targets' intermediacy and influenceability, the persuaders' impact and the network size.

Proposition 5. *Let $\lambda = \gamma = \mu$. If (i, j, k) with $j \notin \{i, k\}$ is a pure strategy Nash equilibrium of the game $\mathcal{G}_\lambda$, then $\tilde{x}_N(i, j, k) = \frac{n}{2}$, $\pi^A_\lambda(i, j, k) = \pi^C_\lambda(i, j, k) = \frac{1}{4}$, $\pi^B_\lambda(i, j, k) = 0$ and, in particular, the following two conditions hold:*

$$b^j_k - 2b^k_j \leq \frac{n}{\lambda}\left(d_j c^k_j - d_k c^j_k\right) \tag{4.19}$$

$$b^j_i - 2b^i_j \leq \frac{n}{\lambda}\left(d_j c^i_j - d_i c^j_i\right) \tag{4.20}$$

Note that when $i = k$, conditions (4.19) and (4.20) coincide and determine a necessary condition for the Nash equilibrium (i, j, i).

According to Proposition 5, if (i, j, k) is a Nash equilibrium, then the difference between the intermediacy of j over k (and over i, respectively) and the double intermediacy of k (and of i, respectively) over j is not greater than the difference between the influenceability of j and the influenceability of k (and of i, respectively). From Theorem 2 and Proposition 5 it follows that, in particular, if (j, j, j) and (i, j, k) are both Nash equilibria of the game, then we must have the equalities in (4.19) and (4.20).

Below we examine some small size societies and check if such non-symmetric Nash equilibria happen to exist.

Example 2 (continued)**.** Let us return to a perfectly symmetric society and consider its particular case, a society with $n = 3$ organized by the complete (directed) network, where everybody listens equally to everybody. Suppose that three persuaders with the impacts $\lambda = \gamma = \mu = 1$ and the opinions 1, $\frac{1}{2}$ and 0 enter the society. As shown in Example 2, (i, i, i) is not an equilibrium of such $\mathcal{G}_\lambda$. However, the game has twelve non-symmetric Nash equilibria. These are all six strategy profiles (i, j, i) with $i \neq j$ that lead to the consensus $\frac{1}{2}$, and six profiles (i, j, k) with $i \neq j \neq k$ and $i \neq k$ for which $\tilde{x}_N(i, j, k) = \frac{3}{2}$ but the individual opinions are different from each other. Note that for $n = 4$, i.e., when a society grows but remains structured by a similar (directed) complete network, there is no symmetric Nash equilibrium, but the game does possess the non-symmetric Nash equilibria of the same type.

Example 3 (continued)**.** Consider now a society structured by a star with $n = 3$ and individual 2 being its center, and with three persuaders having the impact $\lambda = \gamma = \mu = 1$. The star listens equally to all peripheral individuals and those listen only to the star. From Example 3 it follows that there exists a symmetric equilibrium $(2, 2, 2)$ in such a network, where every persuader targets the center of the star. It appears that apart from this symmetric equilibrium, the game has sixteen non-symmetric Nash equilibria. More precisely, these are six profiles (i, j, i) with $i \neq j$, and six profiles (i, j, k) with $i \neq j \neq k$ and $i \neq k$ as in the complete network, but also four other Nash equilibria in which two "neighbouring" persuaders target the center, i.e., the profiles $(1, 2, 2)$, $(2, 2, 1)$, $(2, 2, 3)$ and $(3, 2, 2)$. Again, the equilibria (i, j, i) lead to the consensus $\frac{1}{2}$, while (i, j, k) are such that $\tilde{x}_N(i, j, k) = \frac{3}{2}$ but do not lead to a consensus.

When such a society gains one more peripheral individual, obviously the game has always one symmetric equilibrium $(2, 2, 2)$, but only three non-symmetric Nash equilibria in which the extreme persuaders target the center of the star and the centrist persuader chooses one of the periphery individual. In other words, the non-symmetric equilibria are of the form $(2, 1, 2)$, $(2, 3, 2)$, and $(2, 4, 2)$, and all pure strategy Nash equilibria lead to the consensus $\frac{1}{2}$ and $\tilde{x}_N(i, j, k) = 2$.

Example 4 (continued)**.** In Example 2 we have seen the game which does not have any symmetric equilibrium but does admit non-symmetric Nash equilibria. It is also possible that both symmetric and non-symmetric Nash equilibria exist as presented in Example 3. Now consider a four-individual society interacted in a directed circle, where every individual listens to the next one, and only to him. As shown in Example 4, the game does not possess any symmetric equilibrium, and one can show that in this case non-symmetric equilibria do not exist either. Hence, this game does not have any pure strategy Nash equilibrium.

Example 5 (continued)**.** In a society with four individuals organized in a line network, such that $d_1 = d_4 = 1$ and $d_2 = d_3 = 2$, symmetric Nash equilibria do not exist (see Example 5), but the game admits four non-symmetric Nash equilibria: $(2, 3, 2)$, $(3, 2, 3)$, $(2, 4, 2)$ and $(3, 1, 3)$, all leading to the consensus $\frac{1}{2}$. In other words, under equilibrium the extreme persuaders target one of the "middle" individuals while the centrist persuader chooses either another "middle" individual or the "end" individual which is not the neighbour of the extreme persuaders' target.

Unequal persuasion impacts Next we briefly discuss the case of unequal impacts of the persuaders. According to Theorem 1, at each symmetric strategy profile (i, i, i) the payoffs are given by (4.6) and the aggregate opinion by (4.2). Assume that $\lambda > \gamma > \mu$. It is clear that as the persuasion impact λ of A increases, the aggregate opinion $\tilde{x}_N(i, i, i)$ gets closer to n. It means that all the influence in the network is going under control of persuader A. In such a situation persuaders B and C have to conceal their intentions in order to keep their fraction of influence among the individuals in the society. In this case, they are using mixed strategies.

As in the framework of Grabisch et al. (2018), if the impact levels $\gamma, \mu > 0$ of persuaders B and C are fixed and the impact λ of persuader A is sufficiently large, then the game $\mathcal{G}_{\lambda,\gamma,\mu}$ has only equilibria in mixed strategies.

When the impact of persuader A tends towards infinity, then:

$$\lim_{\lambda\to+\infty} \tilde{x}_N(k, i, i) = \frac{2\big((\gamma + \mu)b_i^k + d_i c_i^k n\big) + \gamma b_k^i}{2(d_i c_i^k + \gamma + \mu)} = n\left(\frac{(\gamma + 2\mu)\frac{b_i^k}{n} + \gamma + 2d_i c_i^k}{2(d_i c_i^k + \gamma + \mu)}\right) \tag{4.21}$$

When λ, μ are fixed and persuader B has the infinite impact γ, then:

$$\lim_{\gamma\to+\infty} \tilde{x}_N(i, k, i) = \frac{2\lambda n - (\lambda - \mu)b_i^k + d_i c_i^k n}{2(d_i c_i^k + \lambda + \mu)} = \frac{n}{2}\left(\frac{2\lambda - (\lambda - \mu)\frac{b_i^k}{n} + d_i c_i^k}{d_i c_i^k + \lambda + \mu}\right) \tag{4.22}$$

and for persuader C with very large μ, when the impacts of A and C are fixed:

$$\lim_{\mu\to+\infty} \tilde{x}_N(i, i, k) = \frac{(2\lambda + \gamma)n - (2\lambda + \gamma)b_i^k}{2(d_i c_i^k + \lambda + \gamma)} = n\left(\frac{(2\lambda + \gamma)(1 - \frac{b_i^k}{n})}{2(d_i c_i^k + \lambda + \gamma)}\right) \tag{4.23}$$

(4.21), (4.22) and (4.23) determine the aggregate opinions when one of the persuaders dominates by exerting infinite impact while the others have fixed impacts and target the same individual. We have $\frac{b_i^k}{n} < 1$. Persuader A wants the aggregate opinion to be as close as possible to the total number of individuals in the society, persuader B – as close as possible to $\frac{n}{2}$, and persuader C aims at having the aggregate opinion as close as possible to 0. It follows that for the persuader with very high impact the aggregate opinion becomes closer to the n times the persuader's opinion as $d_i c_i^k$ increases. In all three cases the dominant persuader is better off by not only targeting highly influential individuals but also by reducing the influence that his opponents have on their target, i.e., by preventing the opponents' target to escape from the influence of the dominant persuader.

Example 1 (continued)**.** Next we consider pure strategy Nash equilibria in Example 1 with the society of five individuals interacting according to the directed graph given by Figure 1.

When $\lambda = \gamma = \mu = 1$, then $(2, 2, 2)$ corresponding to the situation 1 examined earlier in the example is a Nash equilibrium of the game. It appears to be the only symmetric Nash equilibrium of the form (i, i, i). Apart from that, the game played by the equally strong companies has four Nash equilibria of the form (i, k, i) for $k \neq i$, and these are the profiles $(2, 1, 2)$, $(2, 3, 2)$, $(2, 4, 2)$ (corresponding to the analyzed situation 2) and $(2, 5, 2)$. There does not exist any Nash equilibrium (i, j, k) with i, j and k being all different and $\lambda = \gamma = \mu = 1$. In other words, under any pure strategy Nash equilibrium of the game, the extreme persuaders (mobile operators A and C) target individual 2 and the centrist persuader (mobile operator B) targets any of the five individuals. For all these Nash equilibrium profiles, the payoffs of the companies A, B and C are equal to $\frac{1}{4}$, 0 and $\frac{1}{4}$, respectively.

When $\lambda = 4$, $\gamma = 3$ and $\mu = 8$ as assumed in the situation 3 examined before, the game has no pure strategy Nash equilibrium.

Pure strategy Nash equilibria in random networks We investigate numerically the existence of pure strategy Nash equilibria in random networks, by testing empirically Erdös-Rényi random graphs and networks generated by a preferential attachment. Our simulations are performed for the increasing size of the population and connectivity. For the construction of Erdös-Rényi networks, we use $n \in \{4, 8, 12, 16, 20, 24, 28, 32, 36, 40\}$ and the connection probability $p \in \{0.09, 0.18, 0.27, 0.36, 0.45, 0.54, 0.65, 0.72, 0.81, 0.9\}$. For networks generated by the preferential attachment process, we have $n \in \{4, 8, 12, 16, 20, 24, 28, 32, 36, 40\}$ and the mean degree $m \in \{2, 4, 6, 8\}$. Furthermore, 1000 networks are generated by 10 independent draws.

$\lambda, \gamma, \mu =$	1, 1, 1	3, 1, 1	5, 1, 1	10, 1, 1
% of equilibria for Erdös-Rényi	0.87	0.57	0.39	0.00
% of equilibria for Preferential Attachment	1	0.35	0.18	0.00

Table 2: Percentage of pure strategy Nash equilibria for games with the growing level λ of persuader A's impact

$\lambda, \gamma, \mu =$	$3, 1, 1$	$3, 3, 1$	$3, 5, 1$	$3, 10, 1$
% of equilibria for Erdös-Rényi	0.57	0.33	0.26	0.00
% of equilibria for Preferential Attachment	0.35	0.21	0.12	0.00

Table 3: Percentage of pure strategy Nash equilibria for games with the growing level γ of persuader B's impact

Table 2 presents the percentage of the cases when we have pure strategy Nash equilibria with increasing λ. When the impacts of all three persuaders are equal, we have the highest number of Nash equilibria. Moreover, networks generated by the preferential attachment give the maximum number, while Erdös - Rényi networks allow for no pure Nash equilibria. There is also a smooth fall of the Nash equilibrium percentage under the increasing impact of one persuader. Finally, in both networks the percentage reaches zero with λ being very high compared to γ and μ.

Furthermore, we study the effect of the increasing impact of the centrist persuader. We consider the case where the extreme persuaders also hold the asymmetric impacts. Table 3 shows that the fraction of pure strategy Nash equilibria is significantly smaller than in the previous case. We still observe a decreasing trend of the number of pure strategy Nash equilibria which vanishes with one persuader being much stronger compared to the others.

The results give us the idea of how the game with three persuaders holding different impacts can evolve. The empirical investigations support our theoretical results and confirm that the emergence of Nash equilibria in pure strategies is possible with the equal impacts. However, the different network structures show that even for the symmetric impacts, the emergence of pure strategy Nash equilibria can be under some conditions. Moreover, the number of these equilibria decreases significantly when there is only one persuader getting stronger.

5 Related literature

There is a vast literature on social networks devoted to modeling and analyzing opinion formation and diffusion; for surveys, see e.g. Acemoglu et al. (2011), Bramoullé et al. (2016), Jackson (2008). A society is usually described as a network whose nodes represent the individuals and the edges represent their social relations. Each individual has an opinion on a certain issue. The opinion can be a binary variable (or vector) which is a good description for a variety of situations (e.g., Clifford (1973), Förster et al. (2013), Glauber (1963)). However, in some cases, e.g. concerning political issues, a continuous variable might be more appropriate for representing an opinion (e.g., DeGroot (1974), Grabisch et al. (2011), Hegselmann et al. (2002)). The updating of individuals' opinions can be based on various rules, e.g., by taking into account opinions of neighbours. Moreover, independently of the opinion updating rule, different approaches to opinion diffusion in a society can be used. For instance, diffusion of opinion can accelerate when opinion leaders or key players are engaged (Borgatti et al. (2009)). Opinions can also be led by informed agents, since finding the opinion leaders needs global knowledge about the topology of the network (Afshar et al. (2010)). A phenomenon closely related to influence and opinion conformity is that of persuasion, which can attempt to influence a person's beliefs, attitudes, intentions, motivations, or behaviors; for surveys and different persuasion methods, see e.g.

Cialdini (2007), Gass et al. (2010). Our paper also contributes to the literature on consensus reaching, the topic studied extensively in different scientific fields; see e.g., French (1981), and Herrera-Viedma et al. (2011) for a survey.

There are essentially two methods of modeling social learning through networks: Bayesian learning, where agents use Bayes' rule to assess the state of the world (e.g., Acemoglu et al. (2011), Bala et al. (1998), Gale et al. (2003)) and non-Bayesian approach, like imitation models, where agents instead consider a weighted average of their neighbours' opinions or actions in a previous period (e.g., DeGroot (1974), Golub et al. (2010)); see e.g. Acemoglu et al. (2011) for a survey. The DeGroot model is such an imitation framework: it involves repeated communication, where people can keep talking to each other and taking weighted averages of information that they get from their friends. There exist various extensions of DeGroot (1974), e.g., works with the updating varying in time and circumstances (e.g., DeMarzo et al. (2003), Friedkin et al. (1990), Friedkin et al. (1990), Krauze (2000)) and the misrepresenting own opinions (Büchel et al. (2015)).

The literature closely related to influence and opinion formation is the one concerning targeting. In computer science literature usually an algorithmic perspective is used to study the target selection for the optimal adoption and diffusion of innovation (e.g., Domingos et al. (2001), Kempe et al. (2003), Richardson et al. (2002)). Also in economics and marketing there is a growing literature that concerns targeting in social networks. Tsakas (2017) studies the optimal targeting strategy in diffusion based on social imitation. Galeotti et al. (2009) model networks in terms of degree distributions and study influence strategies in the presence of local interaction. They consider two groups of agents, where the one group influences the another one, and optimal influence strategies depend of the connectivity of targeted individuals. Yildiz et al. (2013) assume that some agents are "stubborn", i.e., their opinion is fixed at one of the two values. Also Acemoglu et al. (2013) and Acemoglu et al. (2010) analyze an opinion dynamics model with two types of agents: regular, and stubborn or forceful. The competition between firms aiming at maximizing product adoption by consumers located in a social network is also studied e.g. in Bimpikis et al. (2016) and Dubey et al. (2014). Galeotti et al. (2017) propose a framework to examine optimal interventions, when individuals interact strategically with their neighbours. They solve such intervention problems by exploiting the singular value decomposition of strategic interaction matrices.

Opinion formation is crucial for the analysis of voting and political campaigns. In political science there is a well known theory of spatial allocation. Downs (1957) represents the relative positioning of political parties and voters by using a spatial analogy built on the work of Hotelling (1929) that consists in representing the political preferences on a linear scale from left to right. Tsakas et al. (2018) support formal results of Palfrey (1984) by providing experimental evidence of the strategic polarization of two candidates in electoral competition in the context of the unidimensional spatial model when third party entry is expected. Forsythe et al. (1996) consider voters' behaviour in three-candidate elections in a non-spatial context. Pajala et al. (2018) develop a voting advice model to match voters with political candidates, that accounts for political power, media visibility, and proximity of opinions. They apply their model to Parliamentary Elections in Finland. Lever (2010) analyzes the strategic campaign spending in elections by using the network perspective. He considers a framework with two persuaders (political parties, competing lobbies) who allocate resources to sway voters, and shows that the unique pure strategy Nash equilibrium is such that the spending on each voter is proportional to his eigenvector centrality.

The literature that links network centrality with economic outcomes is growing. Ballester et al. (2006) and Candogan et al. (2012) characterize the Nash equilibrium with a player's action being proportional to his Bonacich centrality (Bonacich (1987)). The key player in Ballester et al. (2006) is identified by an intercentrality measure that takes into account both a player's centrality and his contribution to the centrality of others. Tsakas (2014) analyses targeting in the context of viral marketing and shows that the optimal targeting strategy involves the individuals' decay centrality. Also Banerjee et al. (2013) and Banerjee et al. (2017) study the problem of identifying the most influential agents in a process of information transmission. They introduce diffusion centrality which measures how extensively the information spreads from a given player. The best targets in Grabisch et al. (2018) are characterized by another (new) network centrality called intermediacy, which is also the key concept in the present paper.

6 Conclusions

In this paper we studied a model of competitive opinion formation in a social network. Our point of departure was the model of influence Grabisch et al. (2018) with two strategic players having opposite opinions and targeting non-strategic agents in a network. We extended that framework by adding a "centrist persuader" and focused on the effects of the presence of the third persuader on opinion convergence and consensus reaching in the society, on conditions for Nash equilibria in the game played by the persuaders, and on characterizations of targets in the extended model.

Due to the basic assumptions of the DeGroot model, opinion convergence is preserved, although obviously the long run opinion in the society is different from the one reached in the presence of only two persuaders. Furthermore, we showed that consensus can emerge in the society if the three persuaders target the same individual. The study reveals that in this case, if additionally the persuaders are of the equal impact, the centrist persuader has no effect on the social opinion, but the outcome turns out to be the best and ideal for him. The same "middle outcome" is already obtained when only the extreme persuaders target the same individual and have the equal impact, independently of the behaviour and impact of the centrist persuader.

In the presence of the new (centrist) persuader, the game is not constant sum anymore, as it was the case in Grabisch et al. (2018), and the payoffs are defined in a new way. Each persuader aims as having the average opinion in the society as close as possible to his own opinion, and therefore the game is defined as a system of minimization problems. Furthermore, we considered equilibria of the game. While Grabisch et al. (2018) focus their analysis on the symmetric equilibria in the two-persuader framework, we additionally examined the non-symmetric equilibria in the three-persuader model. Our illustrative examples showed that the existence of a pure strategy Nash equilibrium does depend on the structure of the network. We showed that there exist influence networks that admit equilibria in pure strategies, i.e., star networks. This type of networks have an individual with outstanding characteristics that makes it possible to have a symmetric equilibria in pure strategies, but also non-symmetric equilibria. Indeed, there exist structures that admit both symmetric and non-symmetric equilibria. However, a network can have no symmetric equilibrium but can admit non-symmetric ones, and it can also have no pure strategy Nash equilibrium. For example, there is no equilibrium in pure strategies in circular networks. Similarly, no symmetric equilibrium in pure strategies exists in perfectly symmetric networks, but such structures can admit non-symmetric equilibria.

In order to stress differences and similarities between the sample model and the extended framework, we used Grabisch et al. (2018) as a baseline and our results are framed by using some notions and definitions introduced in Grabisch et al. (2018). We showed that a symmetric equilibrium in pure strategies emerges when the persuaders exert an equal impact. We gave a general condition for the existence of the symmetric equilibrium. It is characterized by two features of the targets: their influenceability and centrality. In this respect, a difference between the result obtained in Grabisch et al. (2018) and the one in our framework is that in our model the relative influence of a potential target has to be at least twice higher than the one of any other individual in the network. In other words, the persuaders are demanding higher centrality from the individual they want to target to compensate the impact of the additional persuader. Similarly to Grabisch et al. (2018) influenceability gains importance versus centrality as the size of the network grows or the impact of the persuaders decreases. In the case when the persuasion impacts are unequal, the high-impact persuader aims at ensuring preeminence on the network by increasing his centrality and diminishing the influenceability of his opponents' target. As in Grabisch et al. (2018), the low-impact persuaders seek to keep a minimal level of influence by hiding their target from the opponent's impact, and therefore, they must use mixed strategies. When the persuasion impact is very small, the growing number of influencers does not affect the persuaders' targeting behavior, i.e., the weak persuaders keep the same strategy with the appearance of an additional influencer.

It would be interesting to develop several variations of the model. In the present paper, we assumed that the persuaders target individuals simultaneously. One can investigate a framework in which targets are approached sequentially. In such a case, strategies of the second/third mover would define the whole targeting plan for each target of the persuaders moving before him. In the sequential version, one can also impose a restriction that an individual can be targeted by only one persuader, which would reduce such sets of strategies for the second/third mover.

In the present work, we focused on the opinion convergence and consensus reaching in the extended DeGroot model with targeting by three persuaders. We also examined the conditions for the existence of pure strategy Nash equilibria in the game played by the persuaders. In the follow up research, one can focus on mixed strategies, in particular, for the case when no pure strategy Nash equilibrium exists.

Also the extension to the multi-target case would bring a number of new research questions. Each persuader can have a budget, i.e., some amount of impact that he can possibly split among different targets. It would be interesting to see under which conditions the persuader focuses on one target, despite the freedom to allocate his budget among an arbitrary number of individuals. An optimal strategy would not only depend on the network structure and the persuaders' impacts, but also on the way how the persuaders allocate their budget.

7 Appendix

7.1 Appendix A

7.1.1 Proof of Proposition 1

We consider a society represented by a directed graph. Due to the assumption that the social network defined by the adjacency matrix W is connected, the convergence of opinions in our targeting model with three persuaders is a direct consequence of the DeGroot model. It means that the only essential classes, such that no arc is going outside, are the persuaders $\{A\}$, $\{B\}$, and $\{C\}$. Consider a steady state vector $\overline{\mathbf{x}}_N$ such that $\mathbf{x}_N(t+1) = \mathbf{x}_N(t) = \overline{\mathbf{x}}_N(\mathbf{s})$. From (2.6) we have

$$\overline{\mathbf{x}}_N(\mathbf{s}) = \Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s})\begin{bmatrix}1 & \frac{1}{2} & 0\end{bmatrix}' + \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W\overline{\mathbf{x}}_N(\mathbf{s})$$

$$[I - \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W]\,\overline{\mathbf{x}}_N(\mathbf{s}) = \Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s})\begin{bmatrix}1 & \frac{1}{2} & 0\end{bmatrix}'$$

$$\overline{\mathbf{x}}_N(\mathbf{s}) = [I - \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W]^{-1}\Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s})\begin{bmatrix}1 & \frac{1}{2} & 0\end{bmatrix}'$$

$$\Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s})\begin{bmatrix}1 & \frac{1}{2} & 0\end{bmatrix}' = \Delta_{\lambda,\gamma,\mu}(\mathbf{s})\begin{bmatrix}\frac{\lambda}{d_{s_A}}e_{s_A} & \frac{\gamma}{d_{s_B}}e_{s_B} & \frac{\mu}{d_{s_C}}e_{s_C}\end{bmatrix}\begin{bmatrix}1 \\ \frac{1}{2} \\ 0\end{bmatrix}$$

Hence,

$$\overline{\mathbf{x}}_N(\mathbf{s}) = [I - \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W]^{-1}\,\Delta_{\lambda,\gamma,\mu}(\mathbf{s})\left(\frac{\lambda}{d_{s_A}}e_{s_A} + \frac{\gamma}{2d_{s_B}}e_{s_B}\right)$$

$[I - \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W]^{-1}$ is always invertible, and therefore the steady state vector $\overline{\mathbf{x}}_N(\mathbf{s})$ always exists.

7.1.2 Proof of Proposition 2

Let $s_A = s_B = s_C = i$. Hence,

$$\Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s}) = \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ \frac{\lambda}{d_i+\lambda+\gamma+\mu} & \frac{\gamma}{d_i+\lambda+\gamma+\mu} & \frac{\mu}{d_i+\lambda+\gamma+\mu} \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \tag{7.1}$$

where $0 < \frac{\lambda}{d_i+\lambda+\gamma+\mu} + \frac{\gamma}{d_i+\lambda+\gamma+\mu} + \frac{\mu}{d_i+\lambda+\gamma+\mu} \leq 1$.

Note that the solution of $\overline{\mathbf{x}}_N(\mathbf{s}) = \Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s})[1\frac{1}{2}0]' + \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W\overline{\mathbf{x}}_N(\mathbf{s})$ is unique. Let us check if a consensus vector $\overline{\mathbf{x}}_N(\mathbf{s}) = [\alpha \cdots \alpha]'$ is a solution. We have for all rows $j \neq i$:

$$\alpha = 0 + 1 \cdot \alpha = \alpha$$

Since $M_{\lambda,\gamma,\mu}(\mathbf{s})$ is row-stochastic, for i-targeted individual we have:

$$\begin{aligned}
\alpha &= \tfrac{\lambda}{d_i+\lambda+\gamma+\mu} + \tfrac{\gamma}{2(d_i+\lambda+\gamma+\mu)} + \big(1 - \tfrac{\lambda}{d_i+\lambda+\gamma+\mu} - \tfrac{\gamma}{d_i+\lambda+\gamma+\mu} - \tfrac{\mu}{d_i+\lambda+\gamma+\mu}\big)\alpha \\
\alpha &= \tfrac{2\lambda+\gamma}{2(d_i+\lambda+\gamma+\mu)} \tfrac{d_i+\lambda+\gamma+\mu}{\lambda+\gamma+\mu} \\
\alpha &= \tfrac{2\lambda+\gamma}{2(\lambda+\gamma+\mu)}
\end{aligned}$$

7.1.3 Proof of Proposition 3

Suppose $s_A = s_C = i$ and $s_B = k \neq i$. Then

$$\Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s}) = \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ \frac{\lambda}{d_i+\lambda+\mu} & 0 & \frac{\mu}{d_i+\lambda+\mu} \\ 0 & \frac{\gamma}{d_k+\gamma} & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

Since the solution of $\overline{\mathbf{x}}_N(\mathbf{s}) = \Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s})[1\frac{1}{2}0]' + \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W\overline{\mathbf{x}}_N(\mathbf{s})$ is unique, similarly to the previous result, we can check if a consensus vector $\overline{\mathbf{x}}_N(\mathbf{s}) = [\alpha \cdots \alpha]'$ is a solution. We have for all rows $j \neq i$ and $j \neq k$:

$$\alpha = 0 + 1 \cdot \alpha = \alpha$$

Since $M_{\lambda,\gamma,\mu}(\mathbf{s})$ is row-stochastic, for i-targeted individual we get:

$$\begin{aligned}
\alpha &= \tfrac{\lambda}{d_i+\lambda+\mu} + \big(1 - \tfrac{\lambda}{d_i+\lambda+\mu} - \tfrac{\mu}{d_i+\lambda+\mu}\big)\alpha \\
\alpha &= \tfrac{\lambda}{d_i+\lambda+\mu} \tfrac{d_i+\lambda+\mu}{\lambda+\mu} \\
\alpha &= \tfrac{\lambda}{\lambda+\mu}
\end{aligned}$$

and for k individual – the target of the centrist persuader:

$$\begin{aligned}
\alpha &= \tfrac{\gamma}{2(d_k+\gamma)} + \big(1 - \tfrac{\gamma}{d_k+\gamma}\big)\alpha \\
\alpha &= \tfrac{\gamma}{2(d_k+\gamma)} \tfrac{d_k+\gamma}{\gamma} \\
\alpha &= \tfrac{1}{2}
\end{aligned}$$

7.1.4 Proof of Theorem 1

First, we recall from Grabisch et al. (2018) some additional notations and two lemmas (for the proofs of the lemmas, see Grabisch et al. (2018)). Let $\overline{C}_i^k$ be the set of cycles around i that do not pass through k, $\overline{B}_i^k$ be the set of walks starting from any individual $\neq i$ that reach i before going through k, and let $F_{i,k}$ be the set of direct walks from i to k, i.e., the set of walks that start in i, end up in k and do not pass through i nor k in between. Moreover, let:

$$\overline{c}_i^k := \textstyle\sum_{p\in\overline{C}_i^k} w(p), \qquad \overline{b}_i^k := \sum_{p\in\overline{B}_i^k} w(p), \qquad f_{ik} := \sum_{p\in F_{i,k}} w(p) \tag{7.2}$$

We have $c_i^k + \overline{c}_i^k = 1$ for all distinct $i,k \in N$. The corresponding set of walks and measures for k are denoted by $\overline{C}_k^i$, $\overline{B}_k^i$, $F_{k,i}$, $\overline{c}_k^i$, $\overline{b}_k^i$ and $f_{k,i}$.

Lemma A1. *For all distinct $i, k \in N$, one has:*

$$\bar{b}_i^k + \bar{b}_k^i = n - 2, \qquad b_i^k + \bar{b}_i^k = n - 1, \qquad b_i^k + b_k^i = n \tag{7.3}$$

Let Γ_i be the set of cycles around i (i.e., walks that start and finish in i and do not pass through i in between) and $y_i = \sum_{p \in \Gamma_i} w(p)$. Let Φ_i be the set of walks to i that have never passed through i before and $\phi_i := \sum_{p \in \Phi_i} w(p)$.

Lemma A2. *For all $i = 1, \cdots, n$, $y_i = 1$ and $\phi_i = n - 1$.*

From Proposition 1 and (3.2):

$$\overline{\mathbf{x}}_N(\mathbf{s}) = [I - \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W]^{-1} \Delta_{\lambda,\gamma,\mu}(\mathbf{s}) \left(\frac{\lambda}{d_{s_A}} e_{s_A} + \frac{\gamma}{2d_{s_B}} e_{s_B} \right)$$

Using the results of Seneta (2006) about non-negative matrices:

Lemma A3. *Let A be a finite $n \times n$ matrix such that $\lim_{k \to \infty} A^k = \boldsymbol{0}$. Then $[I - A]^{-1}$ exists and*

$$[I - A]^{-1} = \sum_{k=0}^{\infty} A^k$$

with $A^0 = I$,

we can modify the long run opinions in the following way:

$$\overline{\mathbf{x}}_N(\mathbf{s}) = \sum_{m=0}^{\infty} (\Delta_{\lambda,\gamma,\mu}(\mathbf{s}) \cdot W)^m \Delta_{\lambda,\gamma,\mu}(\mathbf{s}) \left(\frac{\lambda}{d_{s_A}} e_{s_A} + \frac{\gamma}{2d_{s_B}} e_{s_B} \right) \tag{7.4}$$

(i) If $s_A = s_B = s_C = i$, i.e., if all three persuaders target the same individual i, then

$$\overline{\mathbf{x}}_N(\mathbf{s}) = \sum_{m=0}^{\infty} (\Delta_{\lambda,\gamma,\mu}(\mathbf{s}) \cdot W)^m \frac{2\lambda + \gamma}{2(d_{s_A} + \lambda + \gamma + \mu)}$$

$P_{k,l}^m$ is the set of walks of length m from k to l in the graph G associated to W. For any walk $p = (i_1, \cdots, i_m)$, $w(p)$ denotes its weight measured according to W. $v_k(p)$ is the number of times the walk p passes through k (without taking into account the departure node). We rewrite (7.4) as a sum representing the influence conveyed through each walk of the network and where each passage through one of the targets is re-weighted in order to account for the influence of the persuaders:

$$\overline{\mathbf{x}}_N(\mathbf{s}) = \left(\sum_{k \in N} \sum_{m=0}^{\infty} \sum_{p \in P_{k,s_A}^m} w(p) \left(\frac{d_{s_A}}{d_{s_A} + \lambda + \gamma + \mu} \right)^{v_{s_A}(p)} \right) \frac{2\lambda + \gamma}{2(d_{s_A} + \lambda + \gamma + \mu)} \tag{7.5}$$

The payoff of a link to individual i depends on the degree to which he is influenceable (the number of outgoing links) and his influence (the weighted and discounted number of walks that pass through the individual).

Assume that all three persuaders target i and let Π_i^k be the set of walks ending in i and having gone k times through i before. We decompose the set of walks ending in i according to their total number of passages in i, and rewrite (7.5) as:

$$\tilde{x}_N(i,i,i) = \left(\sum_{k=0}^{\infty} \sum_{p \in \Pi_i^k} w(p) \left(\frac{d_i}{d_i + \lambda + \gamma + \mu} \right)^k \right) \frac{2\lambda + \gamma}{2(d_i + \lambda + \gamma + \mu)}$$

A walk in Π_i^k consists of k cycles around i and possibly a walk to i. We get:

$$\sum_{p \in \Pi_i^k} w(p) = (y_i)^k (1 + \phi_i)$$

and therefore

$$\tilde{x}_N(i,i,i) = \left(\sum_{k=0}^{\infty} \left(y_i \frac{d_i}{d_i + \lambda + \gamma + \mu} \right)^k (1 + \phi_i) \right) \frac{2\lambda + \gamma}{2(d_i + \lambda + \gamma + \mu)}$$

Consequently, we have:

$$\tilde{x}_N(i,i,i) = \frac{(1 + \phi_i)(2\lambda + \gamma)}{2(d_i + \lambda + \gamma + \mu - y_i d_i)}$$

Using Lemma A2, we get (4.2):

$$\tilde{x}_N(i,i,i) = \frac{n(2\lambda + \gamma)}{2(\lambda + \gamma + \mu)}$$

Hence, given (4.1) the payoffs are:

$$\begin{aligned}
\pi^A_{\lambda,\gamma,\mu}(i,i,i) &= \left(\frac{2\mu + \gamma}{2(\lambda + \gamma + \mu)} \right)^2 \\
\pi^B_{\lambda,\gamma,\mu}(i,i,i) &= \left(\frac{\mu - \lambda}{2(\lambda + \gamma + \mu)} \right)^2 \\
\pi^C_{\lambda,\gamma,\mu}(i,i,i) &= \left(\frac{2\lambda + \gamma}{2(\lambda + \gamma + \mu)} \right)^2
\end{aligned} \tag{7.6}$$

(ii) Next, we consider the case where persuaders A and C target i and persuader B targets k, i.e., we assume that $s_A = s_C = i$ and $s_B = k$. Let us then denote (as in Grabisch et al. (2018)) by ϕ_i^k the sum of weights of the walks to i with each passage through k weighted by $\frac{d_k}{d_k+\gamma}$. Let y_i^k be the sum of weights of walks that cycle around i with each passage through k weighted by $\frac{d_k}{d_k+\gamma}$. We have:

$$\begin{aligned}
\tilde{x}_N(\mathbf{s}) = &\left(\sum_{k \in N} \sum_{m=0}^{\infty} \sum_{p \in P^m_{k,s_A}} w(p) \left(\frac{d_{s_A}}{d_{s_A} + \lambda + \mu} \right)^{v_{s_A}(p)} \right) \left(\frac{\lambda}{d_{s_A} + \lambda + \mu} \right) + \\
&\left(\sum_{k \in N} \sum_{m=0}^{\infty} \sum_{p \in P^m_{k,s_B}} w(p) \left(\frac{d_{s_B}}{d_{s_B} + \gamma} \right)^{v_{s_B}(p)} \right) \left(\frac{\gamma}{2(d_{s_B} + \gamma)} \right)
\end{aligned}$$

$$\tilde{x}_N(i,k,i) = \frac{\lambda(1 + \phi_i^k)}{d_i + \lambda + \mu - y_i^k d_i} + \frac{\gamma(1 + \phi_k^i)}{2(d_k + \gamma - y_k^i d_k)} \tag{7.7}$$

The set of walks to i consists in walks to i not passing through k and the set of walks from k to i preceded by an arbitrary number of cycles around k preceded by a walk to k not passing through i. This leads to:

$$\phi_i^k = \overline{b}_i^k + f_{k,i}\frac{d_k}{d_k+\gamma}\left(\sum_{j=0}^{+\infty}\left(\overline{c}_k^i\frac{d_k}{d_k+\gamma}\right)^j\right)(1+\overline{b}_k^i) = \overline{b}_i^k + f_{k,i}\frac{d_k}{d_k+\gamma-\overline{c}_k^i d_k}(1+\overline{b}_k^i) \quad (7.8)$$

The set of cycles around i consists in the set of cycles around i not passing through k and with the set of walks from k to i preceded by an arbitrary number of cycles around k preceded by a walk from i to k. Hence:

$$y_i^k = \overline{c}_i^k + f_{k,i}\frac{d_k}{d_k+\gamma}\left(\sum_{j=0}^{+\infty}\left(\overline{c}_k^i\frac{d_k}{d_k+\gamma}\right)^j\right)f_{i,k} = \overline{c}_i^k + f_{k,i}f_{i,k}\frac{d_k}{d_k+\gamma-\overline{c}_k^i d_k} \quad (7.9)$$

Similarly:

$$\phi_k^i = \overline{b}_k^i + f_{i,k}\frac{d_i}{d_i+\lambda+\mu}\left(\sum_{j=0}^{+\infty}\left(\overline{c}_i^k\frac{d_i}{d_i+\lambda+\mu}\right)^j\right)(1+\overline{b}_i^k) = \overline{b}_k^i + f_{i,k}\frac{d_i}{d_i+\lambda+\mu-\overline{c}_i^k d_i}(1+\overline{b}_i^k) \quad (7.10)$$

$$y_k^i = \overline{c}_k^i + f_{i,k}\frac{d_i}{d_i+\lambda+\mu}\left(\sum_{j=0}^{+\infty}\left(\overline{c}_i^k\frac{d_i}{d_i+\lambda+\mu}\right)^j\right)f_{k,i} = \overline{c}_k^i + f_{i,k}f_{k,i}\frac{d_i}{d_i+\lambda+\mu-\overline{c}_i^k d_i} \quad (7.11)$$

Plugging these equations into (7.7) leads to:

$$\tilde{x}_N(i,k,i) = \frac{\lambda\left(1+\overline{b}_i^k + f_{k,i}\frac{d_k}{d_k+\gamma-\overline{c}_k^i d_k}(1+\overline{b}_k^i)\right)}{d_i+\lambda+\mu-d_i\left(\overline{c}_i^k + f_{k,i}f_{i,k}\frac{d_k}{d_k+\gamma-\overline{c}_k^i d_k}\right)} + \frac{\gamma\left(1+\overline{b}_k^i + f_{i,k}\frac{d_i}{d_i+\lambda+\mu-\overline{c}_i^k d_i}(1+\overline{b}_i^k)\right)}{2\left(d_k+\gamma-d_k\left(\overline{c}_k^i + f_{i,k}f_{k,i}\frac{d_i}{d_i+\lambda+\mu-\overline{c}_i^k d_i}\right)\right)}$$

$$\tilde{x}_N(i,k,i) = \frac{\lambda\left[(1+\overline{b}_i^k)\gamma + d_k(1-\overline{c}_k^i)\left(1+\overline{b}_i^k + f_{k,i}\frac{1+\overline{b}_k^i}{1-\overline{c}_k^i}\right)\right]}{(d_i+\lambda+\mu)(d_k+\gamma-\overline{c}_k^i d_k) - d_i\left(\overline{c}_i^k\gamma + d_k(1-\overline{c}_k^i)\left(\overline{c}_i^k + f_{k,i}f_{i,k}\frac{1}{1-\overline{c}_k^i}\right)\right)} + \frac{\gamma\left[(1+\overline{b}_k^i)(\lambda+\mu) + d_i(1-\overline{c}_i^k)\left(1+\overline{b}_k^i + f_{i,k}\frac{1+\overline{b}_i^k}{1-\overline{c}_i^k}\right)\right]}{2\left[(d_k+\gamma)(d_i+\lambda+\mu-\overline{c}_i^k d_i) - d_k\left(\overline{c}_k^i(\lambda+\mu) + d_i(1-\overline{c}_i^k)\left(\overline{c}_k^i + f_{i,k}f_{k,i}\frac{1}{1-\overline{c}_i^k}\right)\right)\right]}$$

We have:

$$\phi_i = \overline{b}_i^k + f_{k,i}\left(\sum_{k=0}^{+\infty}(\overline{c}_k^i)^k\right)(1+\overline{b}_k^i) = \overline{b}_i^k + f_{k,i}\frac{1}{1-\overline{c}_k^i}(1+\overline{b}_k^i)$$

$$y_i = \overline{c}_i^k + f_{k,i}\left(\sum_{k=0}^{+\infty}(\overline{c}_k^i)^k\right) f_{i,k} = \overline{c}_i^k + f_{i,k}f_{k,i}\frac{1}{1-\overline{c}_k^i}$$

We can get ϕ_k and y_k in a similar way. Hence, we have:

$$\tilde{x}_N(i,k,i) = \frac{\lambda\left[(1+\overline{b}_i^k)\gamma + d_k(1-\overline{c}_k^i)(1+\phi_i)\right]}{(d_i+\lambda+\mu)(d_k+\gamma-\overline{c}_k^i d_k) - d_i\left(\overline{c}_i^k\gamma + d_k(1-\overline{c}_k^i)y_i\right)} + \frac{\gamma\left[(1+\overline{b}_k^i)(\lambda+\mu) + d_i(1-\overline{c}_i^k)(1+\phi_k)\right]}{2\left[(d_k+\gamma)(d_i+\lambda+\mu-\overline{c}_i^k d_i) - d_k\left(\overline{c}_k^i(\lambda+\mu) + d_i(1-\overline{c}_i^k)y_k\right)\right]} \tag{7.12}$$

According to Lemma A2, one has $y_i = 1$ and $\phi_i = n-1$ for all $i = 1, \cdots, n$:

$$\tilde{x}_N(i,k,i) = \frac{\lambda\left[(1+\overline{b}_i^k)\gamma + d_k(1-\overline{c}_k^i)n\right]}{(d_i+\lambda+\mu)(d_k+\gamma-\overline{c}_k^i d_k) - d_i\left(\overline{c}_i^k\gamma + d_k(1-\overline{c}_k^i)\right)} + \frac{\gamma\left[(1+\overline{b}_k^i)(\lambda+\mu) + d_i(1-\overline{c}_i^k)n\right]}{2\left[(d_k+\gamma)(d_i+\lambda+\mu-\overline{c}_i^k d_i) - d_k\left(\overline{c}_k^i(\lambda+\mu) + d_i(1-\overline{c}_i^k)\right)\right]}$$

$$\tilde{x}_N(i,k,i) = \frac{\lambda\left[\gamma b_k^i + d_k c_k^i n\right]}{\gamma d_i c_i^k + (\lambda+\mu)d_k c_k^i + \gamma(\lambda+\mu)} + \frac{\gamma\left[(\lambda+\mu)b_i^k + d_i c_i^k n\right]}{2\left((\lambda+\mu)d_k c_k^i + \gamma d_i c_i^k + \gamma(\lambda+\mu)\right)} \tag{7.13}$$

By following the same procedure, we can get the expressions for (4.4) and (4.5).

7.1.5 Proof of Fact 1

From (4.2), $\tilde{x}_N(i,i,i) = \frac{n(2\lambda+\gamma)}{2(\lambda+\gamma+\mu)}$. Hence, since $\lambda > 0$, $\gamma > 0$ and $\mu > 0$, we have

$$\frac{\partial\tilde{x}_N(i,i,i)}{\partial\lambda} = \frac{n(2\mu+\gamma)}{2(\lambda+\gamma+\mu)^2} > 0 \;\Leftrightarrow\; 2\mu+\gamma > 0$$

$$\frac{\partial\tilde{x}_N(i,i,i)}{\partial\mu} = -\frac{n(2\lambda+\gamma)}{2(\lambda+\gamma+\mu)^2} < 0 \;\Leftrightarrow\; 2\lambda+\gamma > 0$$

$$\frac{\partial\tilde{x}_N(i,i,i)}{\partial\gamma} = \frac{n(\mu-\lambda)}{2(\lambda+\gamma+\mu)^2} \lesseqqgtr 0 \;\Leftrightarrow\; \mu \lesseqqgtr \lambda$$

7.1.6 Proof of Fact 2

When we apply $b_k^i + b_i^k = n$ to equations (4.3), (4.4) and (4.5), we have the following:

$$\frac{\partial\tilde{x}_N(i,k,i)}{\partial b_k^i} = (\lambda-\mu)\gamma, \quad \frac{\partial\tilde{x}_N(i,k,i)}{\partial b_i^k} = (\mu-\lambda)\gamma$$

$$\frac{\partial\tilde{x}_N(k,i,i)}{\partial b_k^i} = (\gamma-2\mu)\lambda, \quad \frac{\partial\tilde{x}_N(k,i,i)}{\partial b_i^k} = (2\mu-\gamma)\lambda$$

$$\frac{\partial\tilde{x}_N(i,i,k)}{\partial b_k^i} = (2\lambda+\gamma)\mu, \quad \frac{\partial\tilde{x}_N(i,i,k)}{\partial b_i^k} = -(2\lambda+\gamma)\mu$$

7.1.7 Proof of Fact 3

We have the following:

$$\frac{\partial \tilde{x}_N(i,k,i)}{\partial (d_i c_i^k)} = \frac{\partial \tilde{x}_N(i,k,i)}{\partial (d_k c_k^i)} = 0$$

$$\frac{\partial \tilde{x}_N(k,i,i)}{\partial (d_i c_i^k)} = \frac{3\left(n d_k c_k^i + \lambda b_k^i\right)}{2\left(d_i c_i^k + 2 d_k c_k^i + 2\lambda\right)^2} > 0$$

$$\frac{\partial \tilde{x}_N(k,i,i)}{\partial (d_k c_k^i)} = \frac{-3\left(n d_i c_i^k + 2\lambda b_i^k\right)}{2\left(d_i c_i^k + 2 d_k c_k^i + 2\lambda\right)^2} < 0$$

$$\frac{\partial \tilde{x}_N(i,i,k)}{\partial (d_i c_i^k)} = \frac{-3\left(n d_k c_k^i + \lambda b_k^i\right)}{2\left(d_i c_i^k + 2 d_k c_k^i + 2\lambda\right)^2} < 0$$

$$\frac{\partial \tilde{x}_N(i,i,k)}{\partial (d_k c_k^i)} = \frac{3\left(n d_i c_i^k + 2\lambda b_i^k\right)}{2\left(d_i c_i^k + 2 d_k c_k^i + 2\lambda\right)^2} > 0$$

7.1.8 Proof of Theorem 2

According to the definition of Nash equilibrium, (i,i,i) is an equilibrium if and only if no individual has a profitable deviation on his own, that is for all $k \in N$:

$$\begin{cases} \pi_\lambda^A(i,i,i) \leq \pi_\lambda^A(k,i,i) \\ \pi_\lambda^B(i,i,i) \leq \pi_\lambda^B(i,k,i) \\ \pi_\lambda^C(i,i,i) \leq \pi_\lambda^C(i,i,k) \end{cases}$$

$$\begin{cases} \left(1 - \frac{1}{n}\tilde{x}_N(i,i,i)\right)^2 \leq \left(1 - \frac{1}{n}\tilde{x}_N(k,i,i)\right)^2 & (a) \\ \left(\frac{1}{2} - \frac{1}{n}\tilde{x}_N(i,i,i)\right)^2 \leq \left(\frac{1}{2} - \frac{1}{n}\tilde{x}_N(i,k,i)\right)^2 & (b) \\ \left(\frac{1}{n}\tilde{x}_N(i,i,i)\right)^2 \leq \left(\frac{1}{n}\tilde{x}_N(i,i,k)\right)^2 & (c) \end{cases}$$

For condition (b):

$$\frac{1}{4} - \frac{1}{n}\tilde{x}_N(i,i,i) + \frac{1}{n^2}\left(\tilde{x}_N(i,i,i)\right)^2 \leq \frac{1}{4} - \frac{1}{n}\tilde{x}_N(i,k,i) + \frac{1}{n^2}\left(\tilde{x}_N(i,k,i)\right)^2$$

$$\frac{1}{n}\left(\tilde{x}_N(i,i,i) - \tilde{x}_N(i,k,i)\right)\left(\tilde{x}_N(i,i,i) + \tilde{x}_N(i,k,i)\right) \leq \tilde{x}_N(i,i,i) - \tilde{x}_N(i,k,i)$$

We have two possibilities:

1) if $\tilde{x}_N(i,i,i) < \tilde{x}_N(i,k,i)$ then $\frac{1}{n}\tilde{x}_N(i,i,i) \geq 1 - \frac{1}{n}\tilde{x}_N(i,k,i)$

2) if $\tilde{x}_N(i,i,i) > \tilde{x}_N(i,k,i)$ then $\frac{1}{n}\tilde{x}_N(i,i,i) \leq 1 - \frac{1}{n}\tilde{x}_N(i,k,i)$

Hence, the system of inequalities becomes:

$$\begin{cases} \tilde{x}_N(i,i,i) \geq \tilde{x}_N(k,i,i) \\ \begin{cases} \tilde{x}_N(i,i,i) < \tilde{x}_N(i,k,i) \\ \tilde{x}_N(i,i,i) \geq n - \tilde{x}_N(i,k,i) \end{cases} \quad or \quad \begin{cases} \tilde{x}_N(i,i,i) > \tilde{x}_N(i,k,i) \\ \tilde{x}_N(i,i,i) \leq n - \tilde{x}_N(i,k,i) \end{cases} \\ \tilde{x}_N(i,i,i) \leq \tilde{x}_N(i,i,k) \end{cases}$$

For particular case where $\lambda = \gamma = \mu$ and given Theorem 1, we get:

$$\begin{cases} n\left(d_k c_k^i - d_i c_i^k\right) \geq \lambda\left(2b_i^k - b_k^i\right) \\ \begin{cases} 0 \geq 0 \\ 1 \leq 1 \end{cases} \quad or \quad \begin{cases} 0 \leq 0 \\ 1 \geq 1 \end{cases} \\ n\left(d_i c_i^k - d_k c_k^i\right) \leq \lambda\left(b_k^i - 2b_i^k\right) \end{cases}$$

From (b) we can conclude that for the centrist persuader, in case when all persuaders have the equal impact, there is no difference between targeting individual i with other persuaders or choosing a different individual k. We can then omit the systems for the centrist persuader, since they do not play any role. We have:

$$\begin{cases} n\left(d_k c_k^i - d_i c_i^k\right) \geq \lambda\left(2b_i^k - b_k^i\right) \\ n\left(d_i c_i^k - d_k c_k^i\right) \leq \lambda\left(b_k^i - 2b_i^k\right) \end{cases}$$

and the final condition is $\lambda\left(b_k^i - 2b_i^k\right) \geq n\left(d_i c_i^k - d_k c_k^i\right)$.

7.1.9 Proof of Proposition 4

We get the limit results by calculating the limits under $\lambda \to 0$ and $\lambda \to +\infty$ in (4.1), using the results of Theorem 1. Next, we apply the definition of Nash equilibrium and compare the payoffs in (i) and (ii), respectively:

$$\frac{1}{2} \leq \frac{3d_k c_k^i}{2(d_i c_i^k + 2d_k c_k^i)} \quad \Leftrightarrow \quad d_i c_i^k \leq d_k c_k^i$$

$$\frac{1}{2} \leq \frac{3b_k^i}{4n} \quad \Leftrightarrow \quad 2b_i^k \leq b_k^i$$

7.1.10 Proof of Proposition 5

Suppose that (i, j, k) with $j \notin \{i, k\}$ is a pure strategy Nash equilibrium. This implies that, in particular, the following conditions must hold:

$$\begin{cases} \pi_\lambda^A(i,j,k) \le \pi_\lambda^A(j,j,k) \\ \pi_\lambda^A(i,j,k) \le \pi_\lambda^A(k,j,k) \\ \pi_\lambda^B(i,j,k) \le \pi_\lambda^B(i,i,k) \\ \pi_\lambda^B(i,j,k) \le \pi_\lambda^B(i,k,k) \\ \pi_\lambda^C(i,j,k) \le \pi_\lambda^C(i,j,i) \\ \pi_\lambda^C(i,j,k) \le \pi_\lambda^C(i,j,j) \end{cases}$$

and therefore, since $\tilde{x}_N(p,p,p) = \tilde{x}_N(p,r,p) = \frac{n}{2}$ for all $p, r \in N$, we have

$$\begin{cases} \tilde{x}_N(i,j,k) \ge \tilde{x}_N(j,j,k) \\ \tilde{x}_N(i,j,k) \ge \frac{n}{2} \\ \pi_\lambda^B(i,j,k) \le \pi_\lambda^B(i,i,k) \\ \pi_\lambda^B(i,j,k) \le \pi_\lambda^B(i,k,k) \\ \tilde{x}_N(i,j,k) \le \frac{n}{2} \\ \tilde{x}_N(i,j,k) \le \tilde{x}_N(i,j,j) \end{cases}$$

This leads to $\tilde{x}_N(i,j,k) = \frac{n}{2}$, and hence from (4.1) we get immediately $\pi_\lambda^A(i,j,k) = \pi_\lambda^C(i,j,k) = \frac{1}{4}$ and $\pi_\lambda^B(i,j,k) = 0$. The third and forth conditions stated in the above system of inequalities are therefore always satisfied, since $\pi_\lambda^B(i,i,k) \ge 0$ and $\pi_\lambda^B(i,k,k) \ge 0$ for all $i, k \in N$. Hence, we get the following:

$$\begin{cases} \tilde{x}_N(j,j,k) \le \frac{n}{2} \\ \tilde{x}_N(i,j,j) \ge \frac{n}{2} \end{cases}$$

Since $\lambda = \gamma = \mu$, by virtue of (4.4) and (4.5), we get

$$\tilde{x}_N(j,j,k) = \frac{3\left(\lambda b_k^j + d_k c_k^j n\right)}{2\left(d_j c_j^k + 2 d_k c_k^j + 2\lambda\right)}$$

$$\tilde{x}_N(i,j,j) = \frac{2\left(2\lambda b_j^i + d_j c_j^i n\right) + \lambda b_i^j + d_i c_i^j n}{2\left(d_j c_j^i + 2 d_i c_i^j + 2\lambda\right)}$$

and therefore the conditions become

$$\begin{cases} b_k^j - 2 b_j^k \le \frac{n}{\lambda}\left(d_j c_j^k - d_k c_k^j\right) \\ b_i^j - 2 b_j^i \le \frac{n}{\lambda}\left(d_j c_j^i - d_i c_i^j\right) \end{cases}$$

7.2 Appendix B

Extension to a multi-target framework

The model considers three persuaders and a society N of n individuals, where each persuader is assumed to target only one individual. We show that there is convergence of opinions in the society and consensus appears in special cases. Next we want to examine if the results on convergence and consensus are still valid when the number of targeted individuals is increased. Let us assume that the persuaders can choose/target two individuals in N to form a link with in order to influence the formation of opinions in the society. We introduce an additional notation for targeted individuals and put upper indices indicating the number of targets: s_A^1, s_A^2, s_B^1, s_B^2, s_C^1, s_C^2. The strategy vector becomes $\mathbf{s} = \left(s_A^1, s_A^2, s_B^1, s_B^2, s_C^1, s_C^2\right)$. The renormalization matrix $\Delta_{\lambda,\gamma,\mu}(\mathbf{s})$ has now diagonal elements equal to

$$\frac{d_1}{d_1 + \lambda\delta_{1,s_A^l} + \gamma\delta_{1,s_B^l} + \mu\delta_{1,s_C^l}}, \cdots, \frac{d_n}{d_n + \lambda\delta_{n,s_A^l} + \gamma\delta_{n,s_B^l} + \mu\delta_{n,s_C^l}} \tag{7.14}$$

with $\delta_{i,s_j^l} = 1$ if $i = s_j^l$ for all $i \in N$, $s_j^l \in \{s_A^1,\, s_A^2,\, s_B^1,\, s_B^2,\, s_C^1,\, s_C^2\}$ and 0 otherwise. The matrix of strategic influence is modified to

$$E_{\lambda,\gamma,\mu}(\mathbf{s}) = \begin{bmatrix} \frac{\lambda}{d_{s_A^1}} e_{s_A^1} & \frac{\gamma}{d_{s_B^1}} e_{s_B^1} & \frac{\mu}{d_{s_C^1}} e_{s_C^1} \\ \frac{\lambda}{d_{s_A^2}} e_{s_A^2} & \frac{\gamma}{d_{s_B^2}} e_{s_B^2} & \frac{\mu}{d_{s_C^2}} e_{s_C^2} \end{bmatrix} \tag{7.15}$$

where e_i^l denotes the unit vector with coordinate 1 at i and j. All other parameters stay the same. Following the same technique as in the proof of Proposition 1, we get the result on opinion convergence.

Fact 4. *For any initial vector of opinions $\boldsymbol{x}(0) := [1 \;\frac{1}{2}\; 0\; \boldsymbol{x}_N(0)]'$, we have*

$$\lim_{t\to+\infty} \left(M_{\lambda,\gamma,\mu}(\boldsymbol{s})\right)^t \begin{bmatrix}1 & \frac{1}{2} & 0 & \boldsymbol{x}_N(0)\end{bmatrix}' = \begin{bmatrix}1 & \frac{1}{2} & 0 & \overline{\boldsymbol{x}}_N(\boldsymbol{s})\end{bmatrix}' \tag{7.16}$$

where

$$\overline{\boldsymbol{x}}_N(\boldsymbol{s}) = [I - \Delta_{\lambda,\gamma,\mu}(\boldsymbol{s})W]^{-1}\Delta_{\lambda,\gamma,\mu}(\boldsymbol{s}) \begin{bmatrix} \frac{\lambda}{d_{s_A^1}} e_{s_A^1} + \frac{\gamma}{2d_{s_B^1}} e_{s_B^1} \\ \frac{\lambda}{d_{s_A^2}} e_{s_A^2} + \frac{\gamma}{2d_{s_B^2}} e_{s_B^2} \end{bmatrix} \tag{7.17}$$

Similarly to Proposition 1, in the model with the increased number of targets there is the opinion convergence in the society. Note that $\overline{\mathbf{x}}_N(\mathbf{s})$ in (7.17) differs from the one in (3.2) by its last part taking into account the second target.[4]

Proposition 2 states that if the three persuaders choose the same target, then the long run opinion converges towards consensus. Following the proof of Proposition 2, we get the same result for the multi-target case.

[4] In order to avoid any confusion, we want to recall that e_s stands for a unit vector with coordinate 1 at s. Therefore, the matrix in (7.15) is a $n \times 3$ matrix and the last term in (7.17) is a $n \times 1$ vector where all entries except s^1 and s^2 are equal to zero.

Fact 5. *If* $s_A^1 = s_B^1 = s_C^1$ *and* $s_A^2 = s_B^2 = s_C^2$ *then the individuals in* N *reach a consensus* α *given by*

$$\alpha = \frac{2\lambda + \gamma}{2(\lambda + \gamma + \mu)} \tag{7.18}$$

In particular, if $\lambda = \mu$*, then the consensus is* $\alpha = \frac{1}{2}$.

As in the framework with one target, this consensus depends on the persuaders' impact. As was mentioned before, the society reaches the same opinion independently of the impact of the centrist player B. This result holds independently of the number of the same targets, i.e., when the three persuaders can choose several targets for diffusion of information.

7.3 Appendix C

Targeting different individuals

Theorem 1 presents the aggregate opinion for the cases, where at least two out of three persuaders target the same individual, i.e., the expressions of $\tilde{x}_N(i,i,i)$, $\tilde{x}_N(i,k,i)$, $\tilde{x}_N(i,i,k)$, and $\tilde{x}_N(k,i,i)$. They are determined in terms of the persuaders' impacts, the network size, the individuals' centrality (influence, intermediacy) and influenceability. By virtue of this theorem, conditions under which the game admits symmetric Nash equilibria in pure strategies are provided in Theorem 2. Consequently, the existence of equilibria depends, e.g., on the network structure determined by the individuals' intermediacy and influenceability.

Consider now the case when all three persuaders choose different targets such as $s_A = i, s_B = j, s_C = k$, with $i \neq j \neq k$, $i \neq k$. For determining the aggregate opinion $\tilde{x}_N(i,j,k)$, the individuals' intermediacy and influenceability are not sufficient anymore, and we need to introduce more complex parameters. Let ϕ_i^{jk} stay for the sum of weights of the walks to i with each passage through j and k weighted by $\frac{d_j}{d_j+\gamma}$ and $\frac{d_k}{d_k+\mu}$, respectively. Moreover, let y_i^{jk} be the sum of weights of walks that cycle around i with each passage through j and k weighted by $\frac{d_j}{d_j+\gamma}$ and $\frac{d_k}{d_k+\mu}$, respectively. Analogically for ϕ_j^{ik} and y_j^{ik}. Then we have the following:

$$\tilde{x}_N(\mathbf{s}) = \left(\sum_{k \in N} \sum_{m=0}^{\infty} \sum_{p \in P_{k,s_A}^m} w(p) \left(\frac{d_{s_A}}{d_{s_A} + \lambda} \right)^{v_{s_A}(p)} \right) \left(\frac{\lambda}{d_{s_A} + \lambda} \right) + \left(\sum_{k \in N} \sum_{m=0}^{\infty} \sum_{p \in P_{k,s_B}^m} w(p) \left(\frac{d_{s_B}}{d_{s_B} + \gamma} \right)^{v_{s_B}(p)} \right) \left(\frac{\gamma}{2(d_{s_B} + \gamma)} \right)$$

and therefore

$$\tilde{x}_N(i,j,k) = \frac{\lambda(1 + \phi_i^{jk})}{d_i + \lambda - y_i^{jk} d_i} + \frac{\gamma(1 + \phi_j^{ik})}{2(d_j + \gamma - y_j^{ik} d_j)}$$

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