Continuous opinion dynamics with anti-conformity behavior

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Abstract

We propose an appropriate updating rule of continuous opinions for modeling anti-conformity behavior, defined according to the repelling function, which gives the shift of the opinion based on the current opinion and the reference opinion for an agent. Two models of continuous opinion dynamics (with opinion value on a continuous scale [0,1]) are studied in undirected networks, by introducing the heterogeneity in the sense of conformity and anti-conformity behavior either in nodes or in links. In the first one, the society is composed of both conformist and anti-conformist agents. Conformist agents update their opinions following the DeGroot rule with equal weights. However, anti-conformist agents would like to repel from others, and the repelling level is negatively related to the opinion distance between the anti-conformist and her reference point. No consensus will be reached for any connected network in the presence of anti-conformist agents. Instead, opinions converge to a disagreement or oscillate over time. In the second part, by supposing a signed graph where agents have positive links (+1) with their friends and negative links (-1) with their enemies, agents update their opinion as the sum of the averaged opinion of their friends and repelling value from their enemies. When the network is balanced, i.e., there are two communitarian groups, and each sub-network corresponding to each group is connected and the initial opinion ranges of the two groups are disjoint, the consensus within each group is guaranteed. Both synchronous and asynchronous updating models are discussed in these two parts.

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1 Introduction

We form our opinions on every aspect of our life, from personal interests (e.g., favorite songs/foods), to social norms (e.g., the acceptable behavior in certain circumstances), to economic decisions (e.g., consumption budget, tax rate), and even to political attitudes, etc. Social networks play a crucial role in modeling opinion dynamics as people are constantly interacting and influencing each other. Experimental evidences provided by Galton (1907), Lorge et al. (1958), Hommes et al. (2005) and Yaniv and Milyavsky (2007) demonstrate that the aggregate (such as median and averaged) estimates of a group are very close to the true value.

Even though the amount of models of opinion dynamics is huge, they can be classified into continuous opinion models and discrete opinion models. The group of models of discrete opinion dynamics is mainly applied to cases when there is no compromises in between any two opinions, actions or decisions, while the group of models of continuous opinion dynamics deals with problems in which the opinion of people can be expressed as real numbers (e.g., tax rates, prices, quantitative predictions). The classical and widely used agent-based model of continuous opinion dynamics is the DeGroot model (French Jr (1956), Harary (1959), Harary et al. (1965), DeGroot (1974)), assuming agents update their opinions iteratively as the weighted average of the opinions of their neighbors. The typical behavior of the DeGroot model is the presence of consensus due to the implicit assumption of conformity, while in real life disagreement is also ubiquitous (Abelson (1964)). Out of this consideration, different kinds of variations of the DeGroot model have been proposed. For example, some variations introduce the stubborn agent\(^1\) whose opinion remains unchanged during the iterative pooling process (see Friedkin and Johnsen (1990), Friedkin and Johnsen (1999), Hegselmann and Krause (2015), Masuda (2015)), by introducing an attachment of each agent to its initial opinion; some other variations consider time-varying weight matrices (Lorenz (2005)); some variations considered that agents are only interacting with those who hold opinions close enough to them by introducing confidence bounds (Hegselmann et al. (2002), Weisbuch (2004), Krause (2000), Deffuant et al. (2000), see also the survey on continuous opinion dynamics with bounded confidence Lorenz (2007)); some other variations introduce negative influences, i.e., the element \(w_{ij}\) of the weight matrix \(W\) can be positive or negative, thus \(W\) is no more row-stochastic (Altafini (2012a), Altafini (2012b)). The DeGroot model implies the assumption of conformity since opinions of agents are attracting each other. Introducing negative influences provides a way to model the anti-conformity behavior.

Conformity V.S. Anti-conformity Before the 21st century, most of the models of opinion dynamics made the basic assumption that agents tend to follow the trend (i.e., they are conformist), and the existence of opposite behavior (anti-conformity or counter-conformity) was neglected. Even in the field of psychology, as Jahoda (1959) criticized, conformity was over-emphasized in the psychological literature, and the emphasis obscured the reality of non-conformity or anti-conformity (Hornsey et al. (2003)). The famous experimental study by Asch (1955) showed that agents tend to conform to the wrong judgement of their predecessors even if some of them know already that the judge-

\(^1\)Stubborn agents are also called by physicists as independent agents (Sznajd-Weron et al. (2011), Sznajd-Weron et al. (2014)), inflexibles (Galam and Jacobs (2007)), zealots (Mobilia (2003)).
ment was wrong. A follow-up study (Deutsch and Gerard (1955)) distinguished two forms of social influence that lead to the wrong judgement. While normative social influence drives some agents to behave like majority in order to avoid "social censure", informational social influence explains the conformity behavior in the sense that agents are uncertain about the answer, so they might rely on the judgement of the majority of the society (Hornsey et al. (2003)). This was later supported by Frideres et al. (1971), Terry et al. (2000), Zafar (2011). Motivated by this idea, Buechel et al. (2015) modeled the continuous opinion dynamics by allowing agents to misrepresent opinions in a conforming or anti-conforming way, and furthermore showed that agents' social power is decreasing in the degree of conformity.

The other branch of study on continuous opinion dynamics with anti-conformity behavior (or negative social influence) is based on the notion of co-opetition, which was introduced by Carfi and Schiﬁro (2012) in the study of the Green Economy and then applied to opinion dynamics with negative influences for a better understanding and explaining the disagreement of opinions. In "coopetitive" networks agents can both cooperate and compete, corresponding to the positive and negative influences among agents, respectively (Proskurnikov and Tempo (2018)), i.e., agents are situated in a signed graph where each edge of the graph is assigned a positive sign or a negative sign. Altafini proposed a model of influence with antagonistic interactions based on the theory of structurally balanced network (Altafini (2012b), Altafini (2012a), Harary et al. (1953)). The idea of structural balancedness can be interpreted as the ancient proverb the friend of my enemy is my enemy, the enemy of my enemy is my friend (Schwartz (2010)). The original Altafini model is coincident with the Abelson model with an influence matrix that can have both positive and negative elements. By doing gauge transformation, the structurally balanced network can be transformed into the corresponding nonnegative network sharing the same convergence properties. It was shown that in case of a structurally balanced network (without self-loops), the bipartite consensus can be achieved. However, for a structurally unbalanced and strongly connected network, the consensus value is always the origin, regardless of the initial conditions (Altafini (2012a), Meng et al. (2016)). In a recent paper coauthored by Altafini (Shi et al. (2019)), the authors defined two rules for negative influences: the opposing rule where the opinion of an agent is attracted by the opposite of the opinion of her neighbor via negative links, and the repelling rule where the two agents repel each other instead of being attracted via negative links. However, none of these two rules is appropriate for modeling the anti-conformity behavior for the following reasons. By adopting the opposing rule, the agent is attracted by the opposite of the reference opinion, so whether the agent is conforming or anti-conforming depends on the relative position of the reference opinion to the origin. By adopting the repelling rule, the deviation of the opinion of one agent is decreasing as the opinion distance with her neighbor decreases, i.e., the opinion of an anti-conformist agent will stay unchanged if she has the same opinion value of her reference opinion which is counterintuitive. Moreover, since opinions are defined in \( \mathbb{R} \), as the force of repelling increases, the norm of the opinions tends to infinity as \( t \) goes to infinity. If \( +\infty \) and \( -\infty \) are considered as the two extreme opinions in real life, it implies that the extreme opinions are never reached, which is also counterintuitive. Thus an appropriate updating rule of continuous opinions

\[\text{2The Abelson model is the continuous counterpart of the DeGroot model.}\]
for anti-conformity behavior still needs to be developed, which is one of the aims of the current paper.

Social behavior is described by sociologists in the following three dimensions: (ir-)relevance, (in-)dependence and (anti-)conformity (Willis (1965)). The current paper focuses on the third dimension, and aims to study the continuous opinion dynamics in undirected networks, considering both conformity and anti-conformity behaviors. As defined by Willis (1965), ”conformity is behavior intended to fulfil normative group expectations as these expectations are perceived by the individual”, while ”anti-conformity behavior is directly antithetical to the norm prescription”, in other words, anti-conformity behavior intends to get away from normative group expectations. Starting from these two definitions, we propose a new opinion updating rule for anti-conformity behavior which is defined by the repelling function, meanwhile we adopt the DeGroot updating for conformity behavior. The (anti-)conformity behavior is introduced either in nodes or in links, respectively. On the one hand, each agent is given a fixed behavioral characteristic, i.e., either conformist or anti-conformist, and they will treat all of their neighbors equally, i.e., the nodes are heterogenous and the links are all the same. On the other hand, all agents adopt the same opinion updating rules, but they divide their neighborhood into friends and enemies, i.e., the nodes are all the same and each link is associated with either a positive or a negative weight. Based on this idea, two models of continuous opinion dynamics are proposed. Opinions are assumed to be a real number in the interval [0,1]. 0 and 1 can be considered as the two extreme opinions. The repelling function of an agent is a real-valued function of the current opinion and the reference opinion of the agent, which gives the deviation of the opinion for anti-conformity behavior. The reference opinion of an agent is a baseline that one agent would like to repel. It can be the average opinions of all her neighbors or the average opinions of all her enemies.

The paper is structured as follows. In section 2, two models of continuous opinion dynamics are introduced based on the repelling function, together with a description of synchronous setting and asynchronous setting for opinions updates. The model of opinion dynamics with conformist and anti-conformists is studied in Section 3 while the model of opinion dynamics over signed graphs is studied in Section 4. Both synchronous and asynchronous updating are studied for the two models. Section 5 concludes the paper with some remarks.

2 The model

2.1 Notation

Let \( N = \{1, 2, \ldots, n\} \) be the society of agents situated in a fixed and undirected network \( G = (N, E) \) whose nodes are the agents and \( E \) is the set of edges or links. The neighborhood of agent \( i \) is denoted as \( N_i = \{ j \in N : \{i, j\} \in E \} \) with its cardinality \( |N_i| =: \eta_i \) being the degree of agent \( i \). We consider that \( i \in N \) always holds for each agent \( i \in N \). Opinion of agent \( i \) at time \( t \) is denoted by \( x_i(t) \) which is a real number in \([0,1]\). In case that opinions converge, the steady state opinion vector is denoted as \( \bar{x} = [\bar{x}_1, \ldots, \bar{x}_n] \). Opinions are bounded between 0 and 1, thus we introduce the notation \([x]_0^1\) to denote the
truncated value of \( x \), i.e.,

\[
[x]_0^1 = \begin{cases} 
0, & \text{if } x < 0 \\
 x, & \text{if } 0 \leq x \leq 1 \\
1, & \text{otherwise.}
\end{cases}
\] (1)

The (anti-)conformity behavior is introduced into either nodes or links, respectively, based on the repelling function. In the first model, the society is supposed to contain agents of two different types, i.e., conformist agents and anti-conformist agents. Conformists would like to hold opinions closer to the averaged social opinion, while anti-conformist would like to do the opposite (see detailed explanation in Section 2.3). In the second model, we suppose that homogeneous agents are situated in an undirected signed network \( G = (N, E) \) where links are associated with a positive or negative sign. Agents linked by an edge with positive sign have opinions which are attracting each other (i.e., conforming influence), while agents linked by an edge with negative sign have opinions which are repelling each other (i.e., anti-conformity influence, see detailed explanation in Section 2.4). In the remaining part of the paper, CODA refers to continuous opinion dynamics with anti-conformity, and these two models will be called the CODA-node model and the CODA-link model for short, respectively.

### 2.2 The repelling function \( f_i(x_i, r_i) \)

To depict the anti-conformity behavior, for a given agent \( i \in N \), the repelling function \( f_i \) is defined as a real-valued function of the current opinion \( x_i \) and the reference opinion \( r_i \) in \([0,1] \times [0,1]\). As described in Section 1, anti-conformity is behavior intended to get away from normative group expectations [Willis (1965)]. Here, \( f_i \) is a deviation function referring to the shift of the opinion due to anti-conformity behavior, and the reference opinion \( r_i \) can be seen as the normative group expectations. The reference opinion of agent \( i \) gives the benchmark opinion that agent \( i \) would like to repel. For example, \( r_i \) can be the average opinion of neighbors of anti-conformist agent \( i \) in the CODA-node model (Section 2.3), while \( r_i \) can also be the average opinion of agents from different groups in the CODA-link model (Section 2.4).

We do not give an explicit form of the repelling function \( f_i \), instead, a set of properties are provided as the requirements of \( f_i \) according to the definition of anti-conformity behavior. If we position \( x_i(t) \) and \( r_i(t) \) on the \([0,1]\) axis, and w.l.o.g. assume that \( x_i(t) < r_i(t) \), as shown in Figure 1, the first intuition is that anti-conformity causes \( x_i \) to move to the left, i.e., \( f_i \) and \( x_i - r_i \) have the same sign (property (c)). The second intuition is that as \( r_i(t) \) becomes closer to \( x_i(t) \), the force of anti-conformity is getting stronger, so \( |f_i| \) is decreasing in \( |x_i - r_i| \) (property (a)). The case of \( x_i(t) > r_i(t) \) is symmetric (property (b)).

![Figure 1: The repelling function \( f_i(x_i, r_i) \)](image)

Denoting \( d_i := x_i - r_i \), the opinion distance between agent \( i \) and the reference point
is $|d_i|$. When $|d_i| \geq \epsilon$, $f_i$ is a function of $d_i$, where $\epsilon > 0$ is a small number. Above all, the repelling function $f_i$ should satisfy the following properties:

- When $d_i \in ]-1,-\epsilon[ \cup [\epsilon,1]$, $f_i$ can be written as $f_i(d_i)$, and the following holds:
  
  (a) $|f_i(d_i)|$ is a decreasing function with respect to $|d_i|$;
  (b) $f_i(d_i)$ is symmetric with respect to the origin, i.e., $f_i(-d_i) = -f_i(d_i)$;
  (c) $f_i(d_i)$ and $d_i$ have the same sign, i.e., $f_i(d_i) \cdot d_i \geq 0$;
  (d) $f_i$ is piecewise continuous in the intervals $[-1, -\epsilon]$ and $[\epsilon, 1]$.

- When $d_i \in [-\epsilon, \epsilon]$, i.e., $|x_i - r_i| \leq \epsilon$, the following cases are distinguished:

  (e) $f_i = \begin{cases} 
  \delta_i, & \text{if } 0 \leq x_i < 0.5 - \epsilon, \\
  \alpha, & \text{if } 0.5 - \epsilon \leq x_i \leq 0.5 + \epsilon, \\
  -\delta_i, & \text{otherwise } 0.5 + \epsilon < x_i \leq 1,
  \end{cases}$

  where $\alpha$ is a random variable taking values $\delta$ and $-\delta$ with equal probabilities $\frac{1}{\delta}$.

When $d_i$ is not very small, i.e., $x_i$ is not very close to $r_i$; property (a) says that the magnitude of the deviation of agent $i$ decays as the magnitude of $d_i$ increases, and agent would deviate more as the reference opinion becomes closer; property (c) implies that $f_i$ has the same sign as $d_i$. That is, opinion of agent $i$ will move further away from $r_i$. Property (e) says that if $x_i$ is very close to $r_i$, then the sign of $f_i$, i.e., the direction of the deviation, will depend on the location of $x_i$, and $x_i$ always moves in the direction such that there is more room for the deviation. Here, $\epsilon$ is introduced as an approximation parameter to avoid the unreachable consensus. Take a simple example of the CODA-node model where anti-conformist agent 1 and conformist agent 2 are connected. $f_1$ gives the shift of opinion for anti-conformist agent 1, i.e., $x_1(t+1) = x_1(t) + f_1(x_1, r_1)$. Assume that $x_1(0) < x_2(0)$, and they consider the opinion of each other as the referenced opinion, i.e., $r_1(t) = x_2(t)$, and $r_2(t) = x_1(t)$. Agents update opinions according to $x_1(t+1) = x_1(t) + f_1(x_1(t), r_1(t))$ and $x_2(t+1) = \frac{1}{2}(x_1(t) + x_2(t))$. As time goes by, $x_2(t)$ would approach $x_1(t)$ (i.e., shift to the left) gradually while $x_1(t)$ would deviate from $x_2(t)$ (i.e., also shift to the left). Without introducing this approximation parameter, one would obtain $\lim_{t \to \infty} x_1(t) = \lim_{t \to \infty} x_2(t) = 0$ which is counterintuitive.

$(-\sigma_i, 0), (\sigma_i, 0)$ are the $x$-intercepts of $f_i(d_i)$, i.e., where $f_i$ crosses $x$-axis ($d_i$-axis), with $\sigma_i > 0$ capturing the maximal opinion distance from the referenced point such that agent $i$ is influenced. $[-\sigma_i, \sigma_i]$ is called the repelling interval of agent $i$. If $f_i$ does not cross $x$-axis or $\sigma_i \geq 1$, we will adopt the convention that $\sigma_i = 1$. $\delta_i$ captures the maximum repelling level, i.e., how much is agent $i$ influenced at most. Even though results in the current paper hold for all forms of $f_i$ fulfilling the previous properties, $f_i(d)$ can be convex, linear or concave on $[\epsilon, \sigma_i]$ with respect to $d_i(t)$, depending on the context. Some corresponding examples are given below.

$\alpha$ can also take value $\delta_i$ or $-\delta_i$ to avoid the randomness.

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3
Example 1 (Linear $f_i(d_i)$ when $\epsilon < d_i < \sigma_i$).
Consider the following piecewise linear form of $f_i(d_i)$:

$$f_i = \begin{cases} 
\max(0, \delta_i - \frac{\delta_i}{\sigma_i}d_i), & \text{if } d_i > \epsilon; \\
\min(0, -\delta_i - \frac{\delta_i}{\sigma_i}d_i), & \text{if } d_i < -\epsilon.
\end{cases} \quad (2)$$

When $\epsilon \leq d_i < \sigma_i$, $f_i$ is linear in $d_i$, i.e., the ratio of the changes in $f_i$ and in $d_i$ is fixed.

Example 2 (Convex $f_i(d_i)$ when $\epsilon < d_i < \sigma_i$).
Consider the following convex form of $f_i(d_i)$ when $\epsilon < d_i < \sigma_i$:

$$f_i = \frac{1}{10d_i}, \quad d_i \in [-1, -\epsilon[ \cup ]1, \epsilon]. \quad (3)$$

When $\epsilon < d_i < \sigma_i$, $f_i$ is convex in $d_i$, i.e., the derivative of $f_i$ w.r.t. $d_i$ is increasing with $d_i$.

Example 3 (Concave $f_i(d_i)$ when $\epsilon < d_i < \sigma_i$).
Consider the following concave form of $f_i(d_i)$ when $\epsilon < d_i < \sigma_i$:

$$f_i = \begin{cases} 
\frac{\delta_i}{\sigma_i} \sqrt{\sigma_i^2 - d_i^2}, & \text{if } d_i \in ]\epsilon, 1], \\
-\frac{\delta_i}{\sigma_i} \sqrt{\sigma_i^2 - d_i^2}, & \text{if } d_i \in [-1, -\epsilon[.
\end{cases} \quad (4)$$

When $\epsilon < d_i < \sigma_i$, $f_i$ is concave in $d_i$, i.e., the derivative of $f_i$ w.r.t. $d_i$ is decreasing with $d_i$. 

Figure 2: An example of linear $f_i$ when $\epsilon < d_i < \sigma_i$.
In this section, we consider the model of opinion dynamics with heterogeneous nodes (agents), i.e., with both conformist and anti-conformist agents. Thus, the society $N$ is partitioned into the set of conformist agents $C$ and that of anti-conformist agents $A$, i.e., $N = C \cup A$. Opinion of agent $i$ at time $t$ is denoted by $x_i(t)$ which is a real number in $[0,1]$. Conformist agents update their opinions following the DeGroot rule with equal weights, i.e.,

$$x_i(t+1) = \frac{1}{\eta_i} \sum_{j \in N_i} x_j(t), \forall i \in C.$$  (5)
However, anti-conformist agents would like to deviate from others, i.e., from their reference opinions. The reference opinion of anti-conformist agent $i$ can take different forms. For example, when considering only the synchronous updating of opinions, $r_i(t)$ is defined as the weighted average of opinions of agent $i$’s neighbors, i.e.,

$$r_i(t) = r^S_i(t) := \frac{1}{\eta_i - 1} \sum_{j \in N_i \setminus \{i\}} x_j(t), \forall i \in A. \quad (6)$$

The shift of opinion for anti-conformist agent $i$ is measured by the repelling function $f_i$ defined in Section 2.2. Thus the updating rule followed by anti-conformist agents reads:

$$x_i(t+1) = \left[ x_i(t) + f_i(x_i(t), r_i(t)) \right]_0^1, \forall i \in A. \quad (7)$$

**Example 4.** Consider three agents situated in the following network (Figure 5) with agent 1 being conformist and agents 2, 3 being anti-conformists. The initial opinion vector is $x^a(0) = [0.6, 0.4, 0.9]$, $\sigma_i = \delta_i = 1/2$, $i = 2, 3$, and $\epsilon = 0.001$.

Figure 5: Network Structure with agent 1 being conformist and agent 2, 3 being anti-conformists. The real line at bottom refers to the value of initial opinions of agents.

Conformist agent 1 updates opinion following the DeGroot rule:

$$x_1(t+1) = \frac{1}{3}(x_1(t) + x_2(t) + x_3(t)).$$

For anti-conformist agents 2 and 3, $r^S_2(t) = r^S_3(t) = x_1(t)$, resulting in $d_2(t) = x_2(t) - x_1(t)$ and $d_3(t) = x_3(t) - x_1(t)$. Agents 2 and 3 update opinions according to

$$x_i(t+1) = \left[ x_i(t) + f_i(t) \right]_0^1,$$

where $f_i(t)$ is taking the linear form of the repelling function defined in Example 1 when $i = 2, 3$.

Take agent 2 for example. At time 0, $d_2(0) = x_2(0) - x_1(0) = -0.2$. $f_2(0) = \min(0, -0.5 + 0.2) = -0.3$. Hence $x_2(1) = x_2(0) + f_2(0) = 0.1$ (see Figure 6). However, assuming a different initial opinion of agent 2, say $x'_2(0) = 0.5$ which is still lower than
At time 0

\[ x_2(0) \]  \[ x_2'(0) \]  \[ x_1(0) \]

At time 1

\[ x_2(1) = x_2'(1) \]  \[ x_1(0) \]

Figure 6: Evolution of \( x_2 \) for one period

\( x_1(0) \), this leads to the same opinion of agent 2 at time 1: \( x_2(1) = x_2(0) + \min(0, -0.5 - (x_2(0) - x_1(0))) = 0.1 \). Indeed \( x_2(t + 1) \) only depends on the relative position of \( x_2(t) \) with respect to \( x_1(t) \) and on the value of \( x_1(t) \) since \( x_2(t + 1) = x_2(t) + \min(0, -0.5 - (x_2(t) - x_1(t))) = \min(x_2(t), x_1(t) - 0.5) \).

In general, \( \forall i \in A \),

\[
x_{i(t+1)} = \begin{cases} 
\lfloor \max(x_i(t), x_i(t) + \delta_i - \frac{\delta_i}{\sigma_i}(x_i(t) - r_i(t))) \rfloor_0 & \text{if } x_i(t) - r_i(t) > \epsilon, \\
\lfloor \min(x_i(t), x_i(t) - \delta_i - \frac{\delta_i}{\sigma_i}(x_i(t) - r_i(t))) \rfloor_0 & \text{if } x_i(t) - r_i(t) < -\epsilon, \\
\lfloor x_i(t) + \delta_i \rfloor_0 & \text{if } 0 \leq x_i(t) < 0.5 - \epsilon, x_i(t) \approx r_i(t), \\
\lfloor x_i(t) - \delta_i \rfloor_0 & \text{if } 0.5 - \epsilon \leq x_i(t) \leq 0.5 + \epsilon, x_i(t) \approx r_i(t), \\
\lfloor x_i(t) \rfloor_0 & \text{otherwise.}
\end{cases} 
\]

(8)

When \( \delta_i = \sigma_i \), it reduces to

\[
x_{i(t+1)} = \begin{cases} 
\lfloor \max(x_i(t), r_i(t) + \delta_i) \rfloor_0 & \text{if } x_i(t) - r_i(t) > \epsilon, \\
\lfloor \min(x_i(t), r_i(t) - \delta_i) \rfloor_0 & \text{if } x_i(t) - r_i(t) < -\epsilon, \\
\lfloor x_i(t) + \delta_i \rfloor_0 & \text{if } 0 \leq x_i(t) < 0.5 - \epsilon, x_i(t) \approx r_i(t), \\
\lfloor x_i(t) + \alpha \rfloor_0 & \text{if } 0.5 - \epsilon \leq x_i(t) \leq 0.5 + \epsilon, x_i(t) \approx r_i(t), \\
\lfloor x_i(t) - \delta_i \rfloor_0 & \text{otherwise.}
\end{cases} 
\]

(9)

The opinion value of an anti-conformist agent, say \( i \), at time \( t + 1 \) only depends on the sign of the difference of the current opinion and the reference opinion (i.e., \( \text{sgn}(x_i(t) - r_i(t)) \)) and on the value of the reference opinion (i.e., \( r_i(t) \)) rather than the value of its own current opinion (i.e., \( x_i(t) \)) in case of \( \delta_i = \sigma_i \). This is because \( \sigma_i = \delta_i \) leads to the fact that the slope of \( f_i \) is \(-1\), so each unit increase in \( d_i \) causes one unit decrease in \( f_i \).
The opinions converge to \([0, 0.5, 1]\) as shown in Figure 7a, corresponding to the initial opinion \(x^a(0)\). However, taking a different initial opinion vector equal to \(x^b(0) = [0.4, 0.6, 0.9]\), the opinions oscillate as shown in Figure 7b. In this case, from time 1 on, anti-conformist agent 2 and agent 3 want to be away from agent 2, so they move to the right till reaching value 1; conformist agent 1 also moves to the right due to the conformity behavior. When \(x_1\) becomes close enough to 1, i.e., \(1 - x_1 < \epsilon\), \(x_2\) and \(x_3\) will jump to the value of \(1/2\), as shown in Figure 8a. Again \(x_1\) will also gradually move to 1/2. As \(x_1\) becomes close enough to 1/2, \(x_2\) and \(x_3\) will move to the left till reaching value 0. Under the same reasoning, \(x_1\) will also gradually move to 0 and at some time, \(x_2\) and \(x_3\) will jump to 1/2, as shown in Figure 8b.
(a) Opinion dynamics when $\delta_i = \sigma_i = \frac{1}{2}$, $\epsilon = 0.001$, $x_1(0) = 0.6$, $x_2(0) = 0.4$ and $x_3(0) = 0.9$.

(b) Opinion dynamics when $\delta_i = \sigma_i = \frac{1}{2}$, $\epsilon = 0.001$, $x_1(0) = 0.4$, $x_2(0) = 0.6$ and $x_3(0) = 0.9$.

Figure 7: Opinion dynamics with anti-conformism in Example 4.
Figure 8: Opinion dynamics with anti-conformism in Example 4 when $\delta_i = \sigma_i = \frac{1}{2}$, $\epsilon = 0.001$, $x_1(0) = 0.4$, $x_2(0) = 0.6$ and $x_3(0) = 0.9$. 

(a) Opinion dynamics in 50 time steps.

(b) Opinion dynamics between time steps 700 and 800.
2.4 Heterogeneous links (signed graph)

In this section, we consider the model of opinion dynamics with heterogeneous links (relations), i.e., with both positive and negative links. Thus, \( G = (N, E) \) is a signed graph, and the set of edges \( E \) is partitioned into the set of positive edges \( E^+ \) and the set of negative edges \( E^- \), i.e., \( E = E^+ \cup E^- \). Then the network \( G \) is decomposed into two subnetworks \( G^+ = (N, E^+) \) and \( G^- = (N, E^-) \). The neighborhood of agent \( i \) is partitioned into the set of her friends and the set of her enemies, i.e., \( N_i = N_i^+ \cup N_i^- \), with \( N_i^+ := \{ j \in N_i : g_{ij} > 0 \} \) and \( N_i^- := \{ j \in N_i : g_{ij} < 0 \} \). The opinion of a given agent is updated as the truncated sum of the average opinion of her friends and the deviation from her enemies, i.e.,

\[
x_i(t+1) = \left[ \frac{1}{\eta_i} \sum_{j \in N_i^+} x_j(t) + f(x_i(t), \frac{1}{\eta_i} \sum_{j \in N_i^-} x_j(t)) \right]_0^{1}
\]

where \( f(x_i(t), \frac{1}{\eta_i} \sum_{j \in N_i^-} x_j(t)) \) is the repelling function defined in section 2.2 with \( d_i(t) = x_i(t) - \frac{1}{\eta_i} \sum_{j \in N_i^-} x_j(t) \). We adopt the convention that \( f(x_i(t), \frac{1}{\eta_i} \sum_{j \in N_i^-} x_j(t)) := 0 \) when \( \eta_i = 0 \).

**Example 5.** Consider 3 agents situated in the following network (see Figure 9). Agent 2 and agent 3 are friends and they are enemies to agent 1.

\[
G = \begin{bmatrix}
1 & 2 & 3 \\
1 & -1 & -1 \\
-1 & 1 & 1 \\
-1 & 1 & 1
\end{bmatrix}
\]

Figure 9: A signed graph. Blue edge represents positive influence and red edges represent negative influences.
Agents update opinions according to the following rules:

Agent 1:
\[ x_1(t + 1) = x_1(t) + f(x_1(t), \frac{1}{2}(x_2(t) + x_3(t))) \]

Agent 2:
\[ x_2(t + 1) = \frac{1}{2}(x_2(t) + x_3(t)) + f(x_2(t), x_1(t)) \]

Agent 3:
\[ x_3(t + 1) = \frac{1}{2}(x_2(t) + x_3(t)) + f(x_3(t), x_1(t)) \]

Taking the linear form of the repelling function as in Example 1 and supposing two different initial opinions \( x^a(0) = [0.4, 0.6, 0.9] \) and \( x^b(0) = [0.6, 0.4, 0.9] \), the opinions converge as shown in figure 10a and figure 10b.

Let us compare our model to the model of Shi et al. (2019), presented in Section ??.

**Opposing rule:**

Agent 1:
\[ x_1(t + 1) = x_1(t) - \beta(x_2(t) + x_3(t)) - 2\beta x_1(t) \]

Agent 2:
\[ x_2(t + 1) = x_2(t) + \alpha(x_3(t) - x_2(t)) - \beta(x_1(t) + x_2(t)) \]

Agent 3:
\[ x_3(t + 1) = x_3(t) + \alpha(x_2(t) - x_3(t) - \beta(x_1(t) + x_3(t)) \]

**Repelling rule:**

Agent 1:
\[ x_1(t + 1) = x_1(t) - \beta(x_2(t) + x_3(t)) + 2\beta x_1(t) \]

Agent 2:
\[ x_2(t + 1) = x_2(t) + \alpha(x_3(t) - x_2(t)) - \beta(x_1(t) - x_2(t)) \]

Agent 3:
\[ x_3(t + 1) = x_3(t) + \alpha(x_2(t) - x_3(t) - \beta(x_1(t) - x_3(t)) \]

The corresponding graphs of opinion dynamics are shown in Figure 11a and Figure 11b. By following the opposing rule, agents are attracted by the opinions of their friends and the opposite opinions of their enemies. However, during the second period of Example 3, agent 1 seems to be attracted by her enemies since \( x_1(1) < -x_2(1) \) and \( x_1(1) < -x_3(1) \) (see Figure 11a). Indeed, \( \forall i, j \in N \), agent \( i \) will be attracted by \( x_j \) as long as \( x_j - x_i \) and \( -x_j - x_i \) have the same sign, which is counterintuitive. On the other hand, by following the repelling rule, agents are attracted by the opinions of their friends and repel the opinions of their enemies. As a result, opinion of agent 1 tends to \(-\infty\) and opinions of agent 2 and agent 3 tend to \(\infty\) as \( t \to \infty \)(see Figure 11b). None of these rules are applicable for modelling anti-conformity behavior. However, our updating rule based on the repelling functions is able to capture anti-conformity behavior via negative links, in the sense that agents are attracted by opinions of their friends and repel the opinions of their enemies, and the repelling level is related to the distance between her own opinion and her reference opinion (the average opinion of her neighbors). The repelling level is
decreasing as the distance increases. In Example 5, agents form consensus within each group (see Figure 10).
(a) $\delta_i = \sigma_i = \frac{1}{2}$, $\epsilon = 0.001$, $x_1(0) = 0.4$, $x_2(0) = 0.6$ and $x_3(0) = 0.9$.

Figure 10: Opinion dynamics of Example 5 over signed graphs.

(b) $\delta_i = \sigma_i = \frac{1}{2}$, $\epsilon = 0.001$, $x_1(0) = 0.6$, $x_2(0) = 0.4$ and $x_3(0) = 0.9$.

Figure 10: Opinion dynamics of Example 5 over signed graphs.
(a) Opinion dynamics with opposing rule when $\alpha = \beta = \frac{1}{2}$, $x_1(0) = 0.6$, $x_2(0) = 0.4$ and $x_3(0) = 0.9$.

(b) Opinion dynamics with repelling rule when $\alpha = \beta = \frac{1}{2}$, $x_1(0) = 0.6$, $x_2(0) = 0.4$ and $x_3(0) = 0.9$.

Figure 11: Opinion dynamics of Example 3 over signed graphs with opposing rule and repelling rule.
2.5 Synchronous updating and Asynchronous updating

Different activation regimes (such as synchronization or synchronization of agents’ activation, different interaction size at each time step and so on) can produce different results in opinion dynamic models (Alizadeh et al. (2015)). It may happen that some interesting phenomena exhibited in the synchronous updating model disappear in the asynchronous setting, and therein stability appears instead of striking spatial chaos (Huberman and Glance (1993), Nowak and May (1992)). By describing the order of updates as a sequence of subsets of the population $N$, Bredereck and Elkind (2017) defined the synchronous updating accordingly, as the updating sequence $(N, N, \ldots, N)$ and defined the asynchronous updating as that with each subset being a singleton. This captures the idea that only one agent is active at each time. The active agent can either meet another agent with a certain probability to exchange opinions or observe the opinions of all her neighbors, and thereafter has her own opinion updated.

Acemoglu and Ozdaglar (2011) modeled an asynchronous updating process by supposing that at each time, agent $i$ is chosen to be active with probability $1/n, \forall i \in N$ and in case of agent $i$ being active, agent $i$ will meet agent $j$ and exchange opinions with probability $p_{ij} \geq 0$, where $\sum_{j=1}^{n} p_{ij} = 1, \forall i \in N$. Moreover, for a better approximation of many real situations, some researchers also consider the opinion dynamics in a random neighborhood setting. For example, Grabisch and Li (2020) studied the synchronous opinion dynamics for binary opinions in a random neighborhood setting in which a random neighborhood is realized in each period. Nyczka and Sznajd-Weron (2013) studied the asynchronous $q$-voter model and assumed that both the voter and the group that can influence the voter are randomly chosen (random active agent and random neighborhood). Ramazi et al. (2016) showed that for threshold-based dynamics, the equilibrium can be reached in both the synchronous and asynchronous setting, and it can also be almost surely reached in partial synchronous setting. These results reveal that the asynchrony does not lead to cycles or non-convergence, neither does the irregular network topology. Instead, the coexistence of heterogenous behavior (such as conformity and anti-conformity behavior) play a role in the presence of cycles or non-convergence.

For a given network structure $G = (N, E)$, it is natural to think the following ways to modeling asynchronous updating process. One is to choose an agent at random to be active with probability $1/n$ and the active agent will meet one of her neighbor at random with probability $1/n$. Then the pair of agents $ij$ will exchange their opinions. This is also called randomised gossip model in Boyd et al. (2006), used by Shi et al. (2019) to describe asynchronous random interactions. An alternative way is to choose an agent at random (with probability $1/n$) to be active and the active agent will update her opinion. Take the CODA-node model for example, and assume that agent $i$ is active at time $t$. If $i$ is conformist she will update her opinion as the average opinion of her neighbors, i.e., according to 4 while if she is anti-conformist, she will update her opinion according to

---

4 In partial synchronous updating setting, a random number of agents update opinions simultaneously.
taking the average opinion of her neighbors as the referenced opinion, i.e.,
\[ r_i^A(t) := \frac{1}{n_i - 1} \sum_{j \in N_i \setminus \{i\}} x_j(t), \forall i \in A \]  
(11)

This paper aims to study the anti-conformity behavior, which is related to the response to the average behavior of the society or a group. Therefore, we adopt the latter form of asynchrony where the active agent is able to observe the behavior of the local neighborhood, based on which she will update opinions.

3 Opinion dynamics with conformists and anti-conformists

3.1 Synchronous updating model

In this section, we consider the synchronous updating where agents update opinions simultaneously following rules 5 and 7. We are interested in whether opinions converge, and if so, whether agents will form a consensus in the long run. Define convergence and consensus as follows.

Definition 1 (Opinion convergence). Opinions of agents in set \( N \) are convergent if \( \forall i \in N, \exists x_i^* \in [0, 1] \), such that \( \lim_{t \to \infty} x_i(t) = x_i^* \).

Definition 2 (Consensus). The society is said to reach a consensus if there exists \( x^* \in [0, 1] \), such that \( \lim_{t \to \infty} x_i(t) = x^* , \forall i \in N \).

Different from the classic model of opinion dynamics with only conformist agents, introducing anti-conformist agents into any connected network of conformist agents makes the consensus impossible.

Fact 1. There is no consensus for any connected network with \( A \neq \emptyset \).

Proof. By contradiction, suppose the society will reach a consensus, then \( \exists t \in \mathbb{N} \), such that \( x_i(t) = x^* , \forall i \in N \). Then for any anti-conformist agent \( i \), \( r_i(t) = x^* \).

By property (e) of the repelling function, \( x_i(t+1) \) is either \( x^* + \delta \) or \( x^* - \delta \) which contradicts the definition of consensus.

Consider two connected conformist agents with any initial opinions in \([0, 1]^2\), they will form a consensus on the average of their initial opinions since time 2. However, for the society of two connected anti-conformist agents, the existence of the steady state depends on the value of initial opinions, and the opinions may form a disagreement or oscillations.

Proposition 1. Assume that \( \sigma_i = \sigma, \forall i \in N \). Consider two connected anti-conformist agents with initial opinions \( x_1(0) \) and \( x_2(0) \). Recall that the steady state opinion vector is denoted as \( \bar{x} \).

(i) If \( |x_1(0) - x_2(0)| \geq \sigma \), then \( \bar{x} = [x_1(0), x_2(0)] \) (independence and disagreement);

Remark here that \( r_i^A(t) = r_i^S(t) \). Other forms of the reference opinion can be adopted, depending on the context.
(ii) If $\epsilon < |x_1(0) - x_2(0)| < \sigma$, then $\bar{x} = [x^*, x^{**}]$ and $x^* \neq x^{**}$ (disagreement);

(iii) If $|x_1(0) - x_2(0)| < \epsilon$, $|x_1(0) - 0.5| \geq \epsilon$ and $|x_2(0) - 0.5| \geq \epsilon$, then there is no steady state but an oscillation with period 2, i.e., $\exists t^* \in \mathbb{N}$, such that $\forall t > t^*$, $x_1(t) = x_2(t) = x_1(t + 2)$;

(iv) Otherwise, opinions will almost surely converge to a disagreement.

Proof. $(\sigma_i, 0)$ is the $x$-intercept of $f_i$, and by properties (a) and (c) of $f_i$, we have $f_i(d_i) = 0, \forall d_i \geq \sigma_i$. (It is analogous for $d_i \leq -\sigma_i$.) Then it is easy to check case by case according to the updating rule 7 and the properties of $f_i$.

Example 6. Consider two connected anti-conformist agents with two different initial opinion vectors $x^a(0) = [0.4, 0.4]$ and $x^b(0) = [0.3, 0.5]$. Taking the linear form of the repelling function as in Example 1, the opinions oscillate for the former case and converge to different values for the latter case, as shown in Figure 12 and 13.

Figure 12: Opinion dynamics with two anti-conformists in Example 6. $x_1(0) = x_2(0) = 0.4, \epsilon = 0.001$ and $\delta = \sigma = 0.5$.

If there is one anti-conformist agent connected with one conformist agent, then for any initial opinions in $[0, 1]^2$, the opinions do not converge but oscillate.
Figure 13: Opinion dynamics with two anti-conformists in Example 6. 
$x_1(0) = 0.3, x_2(0) = 0.5, \epsilon = 0.001$ and $\delta = \sigma = 0.5$.

**Proposition 2.**
Consider the society consisting of two connected agents, with agent 1 conformist and agent 2 anti-conformist. Then there will be oscillations instead of a convergence of opinions. And for $x_2$, the oscillations are between 0 and 1.

**Proof.** We show first there is no convergence by contradiction. Suppose the opinions converge, i.e., $\exists \bar{x} = [\bar{x}_1, \bar{x}_2]$ where $\bar{x}_1, \bar{x}_2 \in [0, 1]$ and a time $t_0$, such that for each $t' > t_0$, $x_1(t') = \bar{x}_1$ and $x_2(t') = \bar{x}_2$. Fix a $t' > t_0$, then:

1) if $\bar{x}_1 = \bar{x}_2 < 1/2$, then $x_2(t' + 1) = \min\{1, x_2 + \delta_2\}$ contradicting $x_2(t' + 1) = x_2(t')$;
2) if $\bar{x}_1 = \bar{x}_2 > 1/2$, then $x_2(t' + 1) = \max\{0, x_2 - \delta_2\}$ contradicting $x_2(t' + 1) = x_2(t')$;
3) if $\bar{x}_1 \neq \bar{x}_2$, then $x_1(t' + 1) = \frac{(\bar{x}_1 + \bar{x}_2)}{2}$ contradicting $x_1(t' + 1) = x_1(t')$.

By contradiction, there is no convergence. Then it suffices to show opinions oscillate.

Supposing that at some time $t$, $x_1(t) \leq x_2(t)$, distinguish the following cases:

1) $x_2(t) - x_1(t) < \epsilon$ and $x_2 \leq 1/2 - \epsilon$, then $x_2$ will move to the right, and $x_1$ will also move to the right due to its conformity, till $x_2 = 1$;
2) $x_2(t) - x_1(t) < \epsilon$ and $x_2 \geq 1/2 + \epsilon$, then $x_2$ will move to the left, and $x_1$ will also move to the left due to its conformity, till $x_2 = 0$. As $x_1$ becomes close enough to 0, $x_2$ will jump to the right, followed by $x_1$ moving to the right, till $x_2 = 1$;
3) $x_2(t) - x_1(t) < \epsilon$ and $|x_2 - 1/2| < \epsilon$, then $x_2$ will move either to the right or to the left, and for both cases $x_2$ will eventually reach the value 1;
4) $\epsilon \leq x_2(t) - x_1(t) < \sigma_2$, then $x_2$ will move to the right, and $x_1$ will also move to the right due to its conformity, till $x_2 = 1$;
5) \(x_2(t) - x_1(t) \geq \sigma_2\), then \(x_2(t + 1) = x_2(t)\). However, \(x_1\) will be closer to \(x_2\) as time goes on. Thus there must be a time such that it reduces to case 4).

Above all, there exists a time, such that \(x_2\) reaches value 1. As \(x_1\) will move closer to \(x_2\), there must be a time, such that the difference between \(x_1\) and \(x_2\) is less than \(\epsilon\). Denote by \(t_1\) the smallest time such that \(x_2(t_1) = 1\) and \(x_2(t_1) - x_1(t_1) < \epsilon\).

Analogously, assuming that at some time \(t\), \(x_1(t) \geq x_2(t)\), distinguish the following cases:

1') \(x_1(t) - x_2(t) < \epsilon\) and \(x_2 \geq 1/2 + \epsilon\), then \(x_2\) will move to the left, and \(x_1\) will also move to the left due to its conformity, till \(x_2 = 0\);

2') \(x_1(t) - x_2(t) < \epsilon\) and \(x_2 \leq 1/2 - \epsilon\), then \(x_2\) will move to the right, and \(x_1\) will also move to the right due to its conformity, till \(x_2 = 1\). As \(x_1\) becomes close enough to 1, \(x_2\) will jump to the left, followed by \(x_1\) moving to the left, till \(x_2 = 0\);

3') \(x_1(t) - x_2(t) < \epsilon\) and \(|x_2 - 1/2| < \epsilon\), then \(x_2\) will move either to the right or to the left, and for both cases \(x_2\) will reach the value 0;

4') \(0 < x_1(t) - x_2(t) < \sigma_2\), then \(x_2\) will move to the left, and \(x_1\) will also move to the left due to its conformity, till \(x_2 = 0\);

5') \(x_1(t) - x_2(t) \geq \sigma_2\), then \(x_2(t + 1) = x_2(t)\). However, \(x_1\) will be closer to \(x_2\) as time goes on. Thus there must be a time such that it reduces to case 4).

Above all, there exists a time, such that \(x_2\) reaches value 0. As \(x_1\) will move closer to \(x_2\), there must be a time, such that the difference between \(x_1\) and \(x_2\) is less than \(\epsilon\). Denote by \(t_1'\) the smallest time such that \(x_2(t_1') = 0\) and \(x_1(t_1') - x_2(t_1') < \epsilon\).

Let us take the assumption again that \(x_1(t) \leq x_2(t)\). \(x_2(t_1 + 1) = 1 - \delta, x_1(t_1 + 1) = \frac{x_1(t_1)}{2} + 1/2. x_1\) will move to the left, again trying to be closer to \(x_2\). This is back to the case that \(x_1 \geq x_2\). Denote by \(t_2\) the smallest time such that \(x_2(t_2) = 0\) and \(x_1(t_1) - x_2(t_2) < \epsilon\). Then \(x_2(t_2 + 1) = \delta\) and \(x_1(t_2 + 1) = \frac{x_2(t_2)}{2} + 1/2. x_1\) is back to the case that \(x_1 \leq x_2\), so \(\exists t_3\), such that \(x_2(t_3) = 1\) and \(x_2(t_3) - x_1(t_3) < \epsilon\). This process will be repeated over time. Thus opinions oscillate and there are oscillations between 0 and 1 for \(x_2\). It also holds for the case of \(x_1(t) \geq x_2(t)\) by the same reasoning.

\[\square\]

**Example 7.** Consider two connected agents with agent 1 being anti-conformist and agent 2 being conformist. The initial opinion vector is \(x(0) = [0.4, 0.8]\). Taking the linear form of the repelling function as in Example 1, opinions oscillate as shown in Figure 15.

![Figure 14: Network structure of Example 7](image)

As a consequence, if \(\exists i \in C\) and \(j \in A\) such that \(N_i = N_j = \{i, j\}\), then there will be oscillations instead of a convergence of opinions.

**Fact 2.** Any connected component of conformist agents will form a consensus.\(^6\)

\(^6\) Remark that the notion of a strongly connected component is equivalent to a class (see Definition ??).
Figure 15: Opinion dynamics with one conformist agent and one anti-conformist agent of Example 7.  
\[ x_1(0) = 0.4, \ x_2(0) = 0.8, \ \delta = \sigma = 0.5 \text{ and } \epsilon = 0.001. \]

Now let us consider the case where one anti-conformist agent is connected to a set of connected conformist agents. The following proposition shows that the opinion dynamics do not converge but oscillate.

**Proposition 3.** If there is only one anti-conformist agent, and this agent has at least one link to a set of connected conformist agents, i.e., \( A = \{i\} \), and \( N_i \cap C \neq \emptyset \), then opinions do not converge.

**Proof.** W.l.o.g., assume that \( i = 1 \), so \( A = \{1\} \) and \( C = \{2, \ldots, n\} \).

We show first there is no convergence by contradiction. Suppose the opinions converge, i.e., \( \exists x = [\bar{x}_1, \ldots, \bar{x}_n] \) where \( \bar{x}_1, \ldots, \bar{x}_n \in [0, 1] \) and a time \( t_0 \), such that for each \( t' > t_0 \), \( x_i(t') = \bar{x}_i, \forall i \in N \). Since \( N_i \cap C \neq \emptyset \), we can assume w.l.o.g. that 2 \( \in N_i \cap C \), while \( N_i \cap C \) may also contain other agents. Fix \( t' > t_0 \), then distinguish the following cases.

1) If \( \bar{x}_1 = \bar{x}_2 < 1/2 \), then \( x_1(t' + 1) = \min\{1, x_1 + \delta_1\} \) contradicting \( x_1(t' + 1) = x_1(t') \).  
2) If \( \bar{x}_1 = \bar{x}_2 > 1/2 \), then \( x_1(t' + 1) = \max\{0, x_1 - \delta_1\} \) contradicting \( x_1(t' + 1) = x_1(t') \).  
3) If \( \bar{x}_1 < \bar{x}_2 \), then \( x_2(t' + 1) = \sum_{j \in N_2} (\bar{x}_j) \eta_2 = x_2(t') = \bar{x}_2. \)

There must exist a \( j_1 \in N_2 \) and \( j_1 \neq 1 \), such that \( \bar{x}_2 < \bar{x}_{j_1} \), due to the fact that agent 1 is linked to agent 2 and \( x_1 < x_2 \). Again,

\[ x_{j_1}(t' + 1) = \frac{\sum_{j \in N_{j_1}} (\bar{x}_j) \eta_{j_1}}{\eta_{j_1}} = x_{j_1}(t') = \bar{x}_{j_1}. \]

There must exist a \( j_2 \in N_{j_1} \), such that \( \bar{x}_{j_1} < \bar{x}_{j_2} \). So there is a infinite series of \( j_1, j_2, \ldots \) such that \( \bar{x}_{j_1} < \bar{x}_{j_2} < \ldots \), contradicting the assumption of a finite number of agents.

4) if \( \bar{x}_1 > \bar{x}_2 \), it is analogous to case 3) to get the contradiction. \footnote{Indeed, 3) and 4) imply that in the presence of only one anti-conformist agent, if the opinions converge, then the set of connected conformist agents form a consensus. Otherwise, a contradiction will happen in the same way.}
By contradiction, there is no convergence.

For a society of connected conformist agents, introducing only one anti-conformist agent will break the consensus. Furthermore, if the society is fully connected, there will be oscillations between 0 and 1.

**Proposition 4.** If the network is complete and there is only one anti-conformist agent, say, agent 1, then there will be oscillations instead of a convergence of opinions. And for $x_1$, the oscillations are between 0 and 1.

**Proof.** By Proposition 3 opinions do not converge. Then it suffices to show opinions oscillate. W.l.o.g., assume that $A = \{1\}$ and $C = \{2, \ldots, n\}$. Since the network is complete, $x_i(1) = \sum_{j=1}^{n} x_j(0), \forall i \in C$, i.e., from time 1, all conformist agents form a consensus. Then all conformist agents will have the same behavior after time 1, so we can treat them as one conformist agent. Thus by Proposition 2 there is no convergence but oscillations of opinions. And for $x_1$, i.e., the opinion of the anti-conformist agent, the oscillations are between 0 and 1.

**Example 8.** Consider one anti-conformist agent 1 and a set of connected conformist agents 2, ..., 6, situated in the society with corresponding network structure as shown in Figure 16. The initial opinion vector is $x(0) = [0.4, 0.7, 0.1, 0.16, 0.9, 0.4]$. Take the linear form of the repelling function as in Example 1. As shown in Figure 17 along the dynamics, anti-conformist agent 1 reaches value 0 during first several steps. And this causes agent 2 and agent 4 who are neighbors of agent 1 to decrease their opinion, which will again influence the other conformist agents to decrease their opinions. As opinions of agent 2 and agent 4 become close enough to 0 so that $\frac{(x_2 + x_4)}{2} \leq \epsilon$, $x_1$ will jump to $\delta_1$, which will again cause the other conformist agents to increase their opinions. Opinions oscillate instead of converging, and oscillations are between 0 and 1 for anti-conformist agent 1.

![Figure 16: Network structure of Example 8](image-url)
Figure 17: Opinion dynamics with one conformist agent and one anti-conformist agent of Example 8. \( x(0) = [0.4, 0.7, 0.1, 0.16, 0.9, 0.4] \), \( \delta = \sigma = 0.5 \) and \( \epsilon = 0.001 \).

However, if more than one anti-conformist agents are introduced into the model, even though the consensus is impossible, the convergence of opinion can still be reached under certain conditions. Consider a common situation where anti-conformist agents hold relatively extreme opinions and conformist agents hold relatively mild opinions, Theorem 1 gives the convergence conditions on the initial opinions and the number of neighbours from each group.

**Definition 3** (connected sets). Two disjoint sets of agents \( B \) and \( D \) are said to be connected if each node of one set has at least one neighbor in the other group.

**Theorem 1** (anti-conformists being extremist and conformists being moderate). Suppose that \( A = A_1 \cup A_2 \), where \( A_1, A_2, C \) are non-empty and pairwise connected. Furthermore, \( x_i(0) < x' \leq x_j(0) \leq x'' < x_k(0) \) holds for \( \forall i \in A_1, \forall j \in C \) and \( \forall k \in A_2 \). Let \( \eta_i,A_1, \eta_i,A_2 \) and \( \eta_i,C \) be the number of neighbors of agent \( i \) that belong to sets \( A_1, A_2 \) and \( C \), respectively, and assume that \( \eta_i,A_1 > 0, \eta_i,A_2 > 0, \eta_i,C > 0, \forall i \in N \). If the following inequalities are satisfied for all \( i \in N \):

\[
\frac{x'}{x''} \leq \frac{\eta_i,A_2}{\eta_i,A_1 + \eta_i,A_2} \quad (12)
\]

\[
\frac{1 - x'}{1 - x''} \geq \frac{\eta_i,A_2}{\eta_i,A_1} + 1, \quad (13)
\]

the following will hold:

**Ordering consistency** \( \forall t, \forall i \in A_1, \forall j \in C, \forall k \in A_2, x_i(t) \leq x_j(t) \leq x_k(t) \).
Opinion convergence: \( \forall i \in N, \exists x_i^* \in [0, 1] \), such that \( \lim_{t \to \infty} x_i(t) = x_i^* \).

Steady-state opinions: Denote by \( \bar{x} = [\bar{x}_{A_1}, \bar{x}_{A_2}, \bar{x}_C] \in [0, 1]^N \) the steady-state opinion vector. If \( \sigma_i = 1, \forall i \in N \), then \( \bar{x}_{A_1} = (0, \ldots, 0), \bar{x}_{A_2} = (1, \ldots, 1), \bar{x}_C = [I - Q]^{-1}R_{A_2} \). The matrix \( Q \) and \( R_{A_2} \) can be obtained as follows: define a weight matrix as \( W = (w_{ij}) \) where

\[
\begin{align*}
w_{ij} &= \begin{cases} 1, & \text{if } i, j \in A_1 \cup A_2 \text{ and } i = j \\ \frac{1}{\eta_i}, & \text{if } i \in C \text{ and } g_{ij} = 1 \\ 0, & \text{otherwise}, \end{cases}
\end{align*}
\]

and put \( W \) into the canonical form as

\[
W = \begin{bmatrix} I_{|A_1|} & 0 & 0 \\ 0 & I_{|A_2|} & 0 \\ R_{A_1} & R_{A_2} & Q \end{bmatrix},
\]

where \( I_k \) is the identity matrix of size \( k \).

Proof. We will show first the ordering consistency, i.e., under conditions 12 and 13, \( x_i(t) < x' \leq x_j(t) \leq x'' < x_k(t) \) holds for \( \forall i \in A_1, \forall j \in C, \forall k \in A_2 \), and for all \( t \). It suffices to show this inequality holds for \( t = 1 \) under the given condition. For any conformist agent \( i \), on one hand, \( x_i(1) = \frac{1}{\eta_i} \sum_{j \in N_i} x_j(0) = \frac{1}{\eta_i} (\sum_{j \in N_i, A_1} x_j(0) + \sum_{j \in N_i, C} x_j(0) + \sum_{j \in N_i, A_2} x_j(0)) \geq \frac{1}{\eta_i} (\eta_i, C x' + \eta_{i,A_1} x'' + \eta_{i,A_2}) \geq x' \) by inequality 12 and the fact that \( \eta_i = \eta_i, C + \eta_{i,A_1} + \eta_{i,A_2} \). On the other hand, \( x_i(1) \leq \frac{1}{\eta_i} (\eta_i, C x'' + \eta_{i,A_1} x' + \eta_{i,A_2}) \leq x'' \). The last inequality is guaranteed by inequality 13 and the fact that \( \eta_i = \eta_i, C + \eta_{i,A_1} + \eta_{i,A_2} \).

For any anti-conformist agent \( i \) in \( A_1 \), the reference opinion is

\[
r_i(0) = \frac{1}{\eta_i - 1} \sum_{j \neq i \in N_i} x_j(0) = \frac{1}{\eta_i - 1} (\sum_{j \in N_i, A_1} x_j(0) + \sum_{j \in N_i, C} x_j(0) + \sum_{j \in N_i, A_2} x_j(0)) > \frac{1}{\eta_i - 1} (\eta_i, C x' + \eta_{i,A_2} x'') > x_i(0)
\]

by inequality 12. In case of \( r_i(0) - x_i(0) < \sigma_i \), it will lead to \( x_i(1) < x_i(0) < x' \), and otherwise \( x_i(1) = x_i(0) < x' \).

For any anti-conformist agent \( i \) in \( A_2 \), the referenced opinion

\[
r_i(0) = \frac{1}{\eta_i - 1} \sum_{j \neq i \in N_i} x_j(0) = \frac{1}{\eta_i - 1} (\sum_{j \in N_i, A_1} x_j(0) + \sum_{j \in N_i, C} x_j(0) + \sum_{j \in N_i, A_2} x_j(0)) < \frac{1}{\eta_i - 1} (\eta_{i,A_1} x' + \eta_{i,C} x'' + \eta_{i,A_2} - 1) < x''
\]

by inequality 13. In case of \( x_i(0) - r_i(0) < \sigma_i \), it will lead to \( x_i(1) > x_i(0) > x'' \), and otherwise \( x_i(1) = x_i(0) > x'' \).

Above all, inequalities 12 and 13 are sufficient conditions such that \( x_i(t) \leq x_j(t) \leq x_k(t) \) holds for \( \forall i \in A_1, \forall j \in C, \forall k \in A_2, \forall t \).
The opinion values of agents in set \( A_1 \) (resp., \( A_2 \)) is decreasing (resp., increasing) as time goes to infinity, thus the opinion convergence of anti-conformist agents is guaranteed due to the boundness of opinions.

\( \sigma_i = 1, \forall i \in A \) means that the repelling area is \([0,1]\) for all anti-conformist agents in \( A_1 \) and \( A_2 \). \( \forall i \in A_1, x_i(t) \) is strictly decreasing until it reaches 0 and \( \forall k \in A_2, x_k(t) \) is strictly increasing until it reaches 1. So \( \lim_{t \to \infty} x_i(t) = 0 \) and \( \lim_{t \to \infty} x_k(t) = 1 \). Then this model is equivalent to the DeGroot model with anti-conformist agents being stubborn agents, and the weight matrix is \( W = (w_{ij}) \) where

\[
\begin{cases}
  1, & i, j \in A_1 \cup A_2 \text{ and } i = j, \\
  \frac{1}{\eta_i}, & i \in C \text{ and } g_{ij} = 1, \\
  0, & \text{otherwise}.
\end{cases}
\]

The weight matrix \( W \) can be written into the canonical form (see Appendix A) with two set \( A_1 \) and \( A_2 \) being two essential classes and the set \( C \) being inessential classes as

\[
W = \begin{bmatrix}
I_{|A_1|} & 0 & 0 \\
0 & I_{|A_2|} & 0 \\
R_{A_1} & R_{A_2} & Q
\end{bmatrix}.
\]

By Fact 4 and Lemma 1 in Appendix A, we have \([I - Q]^{-1}\) exists and \([I - Q]^{-1} = \sum_{k=0}^{\infty} Q^k\) with \( Q^0 = I \).

So the steady-state opinion vector of conformist agents \( \bar{x}_C \) must satisfy that \( \bar{x}_C = R_{A_1} \bar{x}_{A_1} + R_{A_2} \bar{x}_{A_2} + Q \bar{x}_C \). By \( \bar{x}_{A_1} = 0 \) and \( \bar{x}_{A_2} = 1 \), we have \( \bar{x}_C = [I - Q]^{-1} R_{A_2} \bar{x}_C \).

**Example 9.** Let \( x' = \frac{1}{3} \) and \( x'' = \frac{2}{3} \), then inequalities 12 and 13 imply that \( \eta_{i,A_1} = \eta_{i,A_2} \). Every agent has exactly the same number of neighbors in \( A_1 \) and \( A_2 \).

**Example 10.** More generally, take \( x' = \frac{1}{q} \) with \( q \in \mathbb{N} \setminus \{1,2\} \) and \( x'' = 1 - x' \). The conditions being

\[
\frac{x'}{x''} \leq \frac{\eta_{i,A_2}}{\eta_{i,A_1} + \eta_{i,A_2}}, \quad \frac{1 - x'}{1 - x''} \geq \frac{\eta_{i,A_2}}{\eta_{i,A_1}} + 1,
\]

they become

\[
\eta_{i,A_1} \geq \frac{\eta_{i,A_2}}{q - 2}, \quad \eta_{i,A_2} \geq \frac{\eta_{i,A_1}}{q - 2}.
\]

Supposing \( \eta_{i,A_1} = k \), we must have \((q - 2)k \geq \eta_{i,A_2}\), i.e., \( \eta_{i,A_2} \) has the form

\[
\eta_{i,A_2} = (q - 2)k - \ell, \quad \ell = 0, \ldots, (q - 2)k.
\]

However, the second condition implies \((q - 2)k - \ell \geq \frac{k}{q - 2}\), which yields to:

\[
\ell \leq \left\lfloor \frac{k(q - 3)(q - 1)}{q - 2} \right\rfloor.
\]
Remarking that
\[
(q - 2)k - \frac{k(q - 3)(q - 1)}{q - 2} = \frac{k}{q - 2},
\]
the final result is: For any node \(i\), its numbers of neighbors in \(A_1, A_2\) must be of the form
\[
\eta_{i,A_1} \in \mathbb{N}, \quad \eta_{i,A_2} = \left\lfloor \frac{\eta_{i,A_1}}{q - 2} \right\rfloor, \ldots, (q - 2)\eta_{i,A_1}.
\]

Example with \(q = 4\): possible couples \((\eta_{i,A_1}, \eta_{i,A_2})\) are
\[
(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), \ldots, (2n, n), \ldots, (2n, 4n), \ldots
\]
In words, the number of neighbors in one group is at most twice and at least half the number of neighbors in the other group. As \(q\) increases, there are more and more possibilities.

Example 10 illustrates that as long as the influence from two extreme anti-conformist groups remain balanced, the ordering consistency will hold, such that extreme anti-conformist agents stay extreme and conformist agents remain moderate.

**Example 11.** \(A_1 = \{1, 2\}, A_2 = \{3, 4\}, C = \{5, 6\}, \delta_i = 0.5, \sigma_i = 1, \forall i \in A\) and \(\epsilon = 0.001\). Consider 6 agents situated in the following network (see Figure 18) with the corresponding matrix equal to
\[
G = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 2 \\
1 & 1 & 1 & 1 & 1 & 3 \\
1 & 0 & 1 & 1 & 0 & 4 \\
1 & 1 & 1 & 0 & 1 & 5 \\
1 & 0 & 0 & 1 & 1 & 6.
\end{bmatrix}
\]

The opinion dynamics are shown in Figure 19 with the limit opinion equal to \(x(\infty) = (0, 0, 1, 1, 4/11, 5/11)\). The weight matrix of the equivalent DeGroot model is
\[
W = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1/5 & 1/5 & 1/5 & 0 & 1/5 & 1/5 \\
1/4 & 0 & 0 & 1/4 & 1/4 & 1/4
\end{bmatrix}
\]
Thus
\[
\begin{align*}
x_5 &= 1/5 + 1/5x_5 + 1/5x_6 \\
x_6 &= 1/4 + 1/4x_5 + 1/4x_6
\end{align*}
\]
which yields

\[
\begin{align*}
x_5 &= \frac{4}{11} \\
x_6 &= \frac{5}{11}.
\end{align*}
\]

Figure 18: Network structure of Example 11

Figure 19: Opinion dynamics with anti-conformists of Example 11

As one can see on Figure 19, the simulation confirms the theoretical result.
3.2 Asynchronous updating model

In this section, at each time, one agent is chosen at random (with probability \(1/n\)) to update her opinion. Assume that agent \(i\) is active at time \(t\). If \(i\) is conformist she will update her opinion as the average opinion of her neighbors, i.e., according to (5), while if she is anti-conformist, she will update her opinion according to (7), taking the average opinion of her neighbors as the referenced opinion, i.e.,

\[
r^A_i(t) := \frac{1}{\eta_i - 1} \sum_{j \in N\setminus\{i\}} x_j(t), \forall i \in A.
\]  

(14)

Similar to the synchronous updating model, if there exist at least one anti-conformist agents in a connected network, then there is almost surely no consensus.

Fact 3. There is almost surely no consensus for any connected network with \(A \neq \emptyset\).

Proof. By contradiction, suppose the society will reach a consensus, then \(\exists t \in \mathbb{N}\), such that \(x_i(t) = x^*, \forall i \in N\). Then for any anti-conformist agent \(i\), \(r_i(t) = x^*\).

Since each agent is chosen with probability \(1/n > 0\), so almost surely there exist a time \(t' > t\), such that an anti-conformist agent \(i \in A\) is chosen to be active. By property (e) of the repelling function, \(x_i(t + 1)\) is either \(x^* + \delta\) or \(x^* - \delta\) which contradicts the definition of consensus.

Recall that in the synchronous updating model, if there are two connected anti-conformist agents \(N = A = \{i, j\}\) holding similar opinions which are not around 0.5, then their opinions will oscillate with period 2; otherwise, they will form a disagreement. However, in the asynchronous updating model, the oscillation will not happen, and the two anti-conformist agents will for sure form a disagreement. Indeed, the oscillation in Proposition 5 is due to the synchronization of the opinions updates.

Proposition 5. Assume that \(\sigma_i = \sigma, \forall i \in N\). Consider two connected anti-conformist agents with initial opinions \(x_1(0)\) and \(x_2(0)\). Regardless of their initial opinions, they will form a disagreement in the end.

Proof. It suffices to illustrate the case when \(|x_1(0) - x_2(0)| < \epsilon\), \(|x_1(0) - 0.5| \geq \epsilon\) and \(|x_2(0) - 0.5| \geq \epsilon\), since for the other cases it will for sure go to a disagreement as in Proposition 1. Suppose agent \(i, i = 1, 2\) is active at time 1, then \(x_i(1)\) will jump with length \(\delta\), and \(x_2(1) = x_2(0)\). Then \(|x_1(1) - x_2(1)| > \epsilon\) falling into one of the other cases, and eventually a disagreement is reached.

If there is one anti-conformist agent connected with one conformist agent, then for any initial opinions in \([0, 1]^2\), the opinions do not converge but oscillate, which is in accordance with the synchronous model. Indeed, the asynchronization does not change the presence but the speed of the oscillations. Indeed, the oscillation in Proposition 2 is not caused by the synchronization of the opinions updates, but by the presence of both conformist and anti-conformist agents.
Proposition 6. Consider the society consisting of two connected agents, with agent 1 conformist and agent 2 anti-conformist. Then there will be oscillations instead of a convergence of opinions. And for $x_2$, the oscillations are between 0 and 1.

Proposition 7. If there is only one anti-conformist agent, and this agent has at least one link to a set of connected conformist agents, i.e., $A = \{i\}$, and $N_i \cap C \neq \emptyset$, then opinions do not converge.

Consider the case of anti-conformists being extremist and conformists being moderate, the same results will be obtained as in the synchronous model. The asynchronization does not change the limit behavior but makes the speed of the convergence slower.

Proposition 8. Theorem 1 holds for asynchronous updating model with active agents observing opinion of all neighbors.

Proof. The techniques used in the proof of Theorem 1 can also be applied to the case of the asynchronous model, since in both synchronous and asynchronous updating model, agents are following the same updating rules, and all the inequalities in the proof of Theorem 1 also hold here.

4 Opinion dynamics over signed graphs

4.1 Synchronous updating model

In this section, we consider the synchronous updating where agents update opinions simultaneously following rules \[10\].

One widely-studied signed graph is the structurally balanced graph (see its general definition in Section \[8\]). For any structurally balanced graphs, the agents can be partitioned into two groups, where agents connect to agents in the same group with positive links and connect to agents in the different group with negative links. So it can also be considered as two communitarian groups. We study first the opinion dynamics for two communitarian groups with disjoint initial opinions.

Definition 4. We say that an undirected network $\mathcal{G} = (N, E)$ is connected, if $\forall i, j \in N$, there is a path from $i$ to $j$, i.e., $i \leftrightarrow j$.

We are interested in under which conditions would opinions of agents converge if there are two communitarian groups with disjoint initial opinions, and will the order of the initial opinions between the two group be consistent over time.

Proposition 9 (Two communitarian groups with disjoint initial opinions).

Consider a society $\mathcal{G} = (N, E)$ composed of two communitarian groups $N = G_1 \cup G_2$, that is, $g_{ij} \in \{1, 0\}$ if $i$ and $j$ are from the same group; $g_{ij} \in \{-1, 0\}$ otherwise. The sub-network $(G_k, E_k)$ is a connected network where $E_k = \{\{i, j\} \in E \mid i \in G_k, j \in G_k\}, k = 1, 2$, i.e., $\forall i, j \in G_k, i \leftrightarrow j, k = 1, 2$. \[8\] Assume that $\sigma_i = \sigma \in (0, 1), \forall i \in N$. If $x_i(0) + \epsilon < x^* < x_j(0) - \epsilon, \forall i \in G_1, \forall j \in G_2$, then the following will hold:

\[8\] Indeed, this is equivalent to the notion of strongly connectedness defined in Section \[??\] (see Definition \[8\]).

\[9\] Note that we do not exclude the case where there is no link between $G_1$ and $G_2$. 

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Ordering consistency
∀t, ∀i ∈ G_1, ∀j ∈ G_2, it holds that x_i(t) < x_j(t).

Opinion convergence and consensus
∀i ∈ G_1, ∀j ∈ G_2, ∃x', x'' ∈ [0, 1] and x' < x'', such that \( \lim_{t \to \infty} x_i(t) = x' \), \( \lim_{t \to \infty} x_j(t) = x'' \).

Steady-state opinion
Moreover, if G_1 and G_2 are connected sets, then at least one of the following cases holds true:
(i) \( x'' - x' \geq \sigma \);
(ii) \( x' = 0 \);
(iii) \( x'' = 1 \).

Proof. ∀i ∈ N, denote the neighborhood of agent i in G_k as N_{i,k} and its cardinality as \( \eta_{i,k} \), where k = 1, 2.

Fix any i ∈ G_1, then
\[
x_i(1) = \left[ \frac{1}{\eta_{i,1}} \sum_{j \in N_{i,1}} x_j(0) + f\left( x_i(0), \frac{1}{\eta_{i,2}} \sum_{j \in N_{i,2}} x_j(0) \right) \right]_0^1.
\]
Remark here that in case of \( \eta_{i,2} = 0 \), we adopt the convention that \( f\left( x_i(0), \frac{1}{\eta_{i,2}} \sum_{j \in N_{i,2}} x_j(0) \right) = 0 \), and that \( \eta_{i,1} \geq 1 \) since \( (G_1, E_1) \) is connected.

By \( x_i(0) + \epsilon < x^* < x_j(0) - \epsilon, \forall i \in G_1, \forall j \in G_2, \frac{1}{\eta_{i,2}} \sum_{j \in N_{i,2}} x_j(0) > x_i(0) + \epsilon \),
thus \( f\left( x_i(0), \frac{1}{|G_2|} \sum_{j \in G_2} x_j(0) \right) \leq 0 \). Moreover, \( \frac{1}{\eta_{i,1}} \sum_{j \in N_{i,1}} x_j(0) < x^* - \epsilon \). Therefore,
\( x_i(1) < x^* - \epsilon, \forall i \in G_1 \).

Analogously, \( x_j(1) > x^* + \epsilon, \forall j \in G_2 \). So \( x_i(t) + \epsilon < x^* < x_j(t) - \epsilon \) holds for any \( t \), i.e., the ordering consistency is satisfied.

∀i ∈ G_1, ∀t, r_i(t) = \[ \frac{1}{\eta_{i,2}} \sum_{j \in N_{i,2}} x_j(t) > x^* + \epsilon > x_i(t) + \epsilon, \]
so \( f(x_i(t), \frac{1}{\eta_{i,2}} \sum_{j \in N_{i,2}} x_j(t) \leq 0 \) always holds. Thus
\[
x_i(t + 1) \leq \frac{1}{\eta_{i,1}} \sum_{j \in N_{i,1}} x_j(t), \forall i \in G_1.
\]
This implies \( \frac{1}{\eta_{i,1}} \sum_{j \in N_{i,1}} x_j(t + 1) \leq \frac{1}{\eta_{i,1}} \sum_{j \in N_{i,1}} x_j(t) \). Therefore the limit of the series \( \{ \frac{1}{\eta_{i,1}} \sum_{j \in N_{i,1}} x_j(t) \}_{t} \) exists (i.e., \( \exists x'_i \), such that \( \lim_{t \to \infty} \frac{1}{\eta_{i,1}} \sum_{j \in N_{i,1}} x_j(t) = x'_i \) due to the weak monotonicity and boundedness. By \( x_i(t + 1) \leq \frac{1}{\eta_{i,1}} \sum_{j \in N_{i,1}} x_j(t), \forall i \in G_1 \), it must hold that \( \lim_{t \to \infty} x_i(t) = x'_i, \forall i \in G_1 \). Analogously, for all \( i \in G_2 \), \( \exists x''_i \) such that \( \lim_{t \to \infty} x_i(t) = x''_i \) and \( x'_i < x''_i, \forall i \in G_1, \forall j \in G_2 \). Till now, the convergence of the opinion vector is guaranteed.
Denote the steady state opinions as $x_i'$ and $x_j''$, $\forall i \in G_1, j \in G_2$, i.e., $\exists^*$, such that $\forall t > t^*, x_i'(t) = x_i'$ and $x_j''(t) = x_j''$.

Then let us show the consensus is reached within each group. Due to the connectedness of each sub-network $(G_k, E_k), k = 1, 2$, it suffices to show that if any two agents $i, j$ of the same group are directly connected, i.e., $i, j \in G_k, g_{ij} = 1, k = 1, 2$, then $x_i' = x_j'$. By contradiction, suppose that $x_i' \neq x_j'$, and w.l.o.g., assume that $x_i' < x_j'$, then $x_j'(t^* + 1) < x_j'(t^*)$ which leads to a contradiction.

To complete the proof, it suffices to show that if $G_1$ and $G_2$ are connected, $x' \neq 0$ and $x'' \neq 1$, then $x'' - x' \geq \sigma$. By contradiction, suppose that $x'' - x' < \sigma$, then for any $i \in G_1$, $\exists j \in G_2$ such that $i$ and $j$ are connected, thus $x_i(t^* + 1)$ will be strictly less than $x_i'(t^*)$ which leads to a contradiction.

\[\Box\]

**Example 12** (Signed graph with two communitarian groups).
Consider two communitarian groups $G_1 = \{1, 2, 9, 10\}$ and $G_2 = \{3, 4, 5, 6, 7, 8\}$ situated in the following network (see Figure 20) with the corresponding matrix equal to

\[
G = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\
\end{bmatrix}
\] (15)

\[
\delta = \sigma = 0.5 \quad \text{and} \quad \epsilon = 0.001. \quad \text{The opinion dynamics are shown in Figure 21. Opinions within each group form a consensus very quickly and there is opinion ordering consistency in the group level. The steady state opinions are } 11/80 \quad \text{and } 3/4 \quad \text{for agents of } G_1 \quad \text{and } G_2, \quad \text{respectively.}
\]

However, all the properties that hold in Propositions may fail when it is generalized to several communitarian groups with disjoint opinions. Let us consider a society $G = (N, E)$ composed of several communitarian groups $N = G_1 \cup \ldots \cup G_k$, where $k > 2$ is an integer, that is, $g_{ij} \in \{1, 0\}$ if $i$ and $j$ are from the same group; $g_{ij} \in \{-1, 0\}$ otherwise. Let us suppose that opinions of agents from distinct groups belong to disjoint opinion ranges, i.e., $\forall i \in G_p, \forall j \in G_{p+1}, x_i(0) + \epsilon < x'_p < x_j(0) - \epsilon, p = 1, \ldots, k - 1$. The following example will give some counter-evidence showing that the ordering consistency for all groups is no more true, and opinions of agents who belong to groups in the middle may oscillate or converge, depending on the values of $\sigma$ and $\delta$. A detailed study of all possible cases seems to be out of reach.

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Example 13 (Signed graph with four communitarian groups).
Consider four communitarian groups $G_1 = \{1, 2\}$, $G_2 = \{3, 4\}$, $G_3 = \{5, 6\}$ and $G_4 =$ \{7, 8, 9, 10\} situated in the following network (see Figure 22) with the corresponding matrix equal to

$$
G = \begin{bmatrix}
1 & 1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\
1 & 1 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & 2 \\
-1 & 0 & 1 & 1 & -1 & 0 & 0 & -1 & -1 & -1 \\
-1 & -1 & 1 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \\
-1 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & -1 & -1 \\
0 & -1 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & -1 \\
-1 & -1 & 0 & -1 & -1 & -1 & 1 & 0 & 1 & 1 \\
0 & -1 & -1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\
-1 & 0 & -1 & -1 & 0 & 1 & 1 & 1 & 1 & 9 \\
0 & -1 & -1 & -1 & -1 & 1 & 0 & 1 & 1 & 1 \\
\end{bmatrix}
$$

(16)
When $\delta = \sigma = 0.5$, the opinion dynamics are shown in Figure 23. Opinions of agents within groups $G_1, G_2$ and $G_4$ form a consensus very quickly, while opinions of agents in group $G_3$ oscillate.

If we fix $\delta = 0.5$, and decrease the value of $\sigma$ to $\sigma = 0.2$, then opinions of agents within every group form a consensus, but the ordering consistency is no more true in this case since $x'_{i} > x'_{j}$, $\forall i \in G_3$, $\forall j \in G_2$ (see Figure 24). If we fix $\delta = 0.5$, and increase the value of $\sigma$ to $\sigma = 0.7$, then opinions of agents within every group form a consensus, and the ordering consistency holds in this case (see Figure 25).

Instead, if we fix $\sigma = 0.5$, and increase the value of $\delta$ to $\delta = 0.9$, then similar to the case of $\delta = \sigma = 0.5$, opinions of agents within groups $G_1, G_2$ and $G_4$ form a consensus very quickly, but opinions of agents in group $G_3$ oscillate with a larger extent (see Figure 26). If we fix $\sigma = 0.5$, and decrease the value of $\delta$ to $\delta = 0.3$, then again opinions of agents within every group form a consensus, but the ordering consistency fails in this case since $x'_{i} > x'_{j}$, $\forall i \in G_1$, $\forall j \in G_2$ (see Figure 27).
4.2 Asynchronous updating model

In this section, at each time, one agent is chosen at random (with probability $\frac{1}{n}$) to update her opinion according to equation \((10)\), taking the average opinion of her neighbors as the referenced opinion, i.e.,

\[
    r_i^A(t) := \frac{1}{\eta_i} \sum_{j \in N_i} x_j(t), \forall i \in A. \tag{17}
\]

Consider the case of two communitarian groups with disjoint initial opinions.

**Proposition 10.** Proposition 9 holds for asynchronous updating model with active agents observing opinion of all neighbors.

**Proof.** The techniques used in the proof of Proposition 9 can also be applied to the case of the asynchronous model, since in both synchronous and asynchronous updating model, agents are following the same updating rules, and all the inequalities in the proof of Proposition 9 also hold here. \(\square\)

**Example 14.**
Consider two communitarian groups $G_1 = \{1, 2, 9, 10\}$ and $G_2 = \{3, 4, 5, 6, 7, 8\}$ situated
Figure 23: Opinion dynamics for two communitarian groups of Example 13 with $\delta = \sigma = 0.5$.

Figure 24: Opinion dynamics for two communitarian groups of Example 13 with $\delta = 0.5$ and $\sigma = 0.2$.

in the same network as in Example 12, $\delta = \sigma = 0.5$ and $\epsilon = 0.001$. Figures 28 and 29 are two realizations of the asynchronous opinion dynamics. Opinions within each group form a consensus very quickly and there is opinion ordering consistency in the group level. However, it is path-dependent and the value of the steady state opinion depends
Figure 25: Opinion dynamics for two communitarian groups of Example 13 with \( \delta = 0.5 \) and \( \sigma = 0.7 \).

Figure 26: Opinion dynamics for two communitarian groups of Example 13 with \( \delta = 0.9 \) and \( \sigma = 0.5 \).

on the activation order of agents. The steady state opinions of agents in \( G_1 \) and \( G_2 \) are \( [0.05, 0.61] \) (corresponding to Figures 28) and 0.2, 0.8 (corresponding to Figures 29), respectively.
Figure 27: Opinion dynamics for two communitarian groups of Example 13, with $\delta = 0.3$ and $\sigma = 0.5$.

Figure 28: Realization 1 of the asynchronous opinion dynamics for two communitarian groups of Example 14, $\delta = \sigma = 0.5$ and $\epsilon = 0.001$. 
Figure 29: Realization 2 of the asynchronous opinion dynamics for two communitarian groups of Example 14. \( \delta = \sigma = 0.5 \) and \( \epsilon = 0.001 \).

Example 15. Consider four communitarian groups \( G_1 = \{1, 2\} \), \( G_2 = \{3, 4\} \), \( G_3 = \{5, 6\} \) and \( G_4 = \{7, 8, 9, 10\} \) situated in the same network as in Example 13. Then the phenomenon of oscillations disappear in the asynchronous updating model. Instead, opinions always converge, regardless of the values of initial opinions and the values of \( \delta \) and \( \sigma \). The dynamic is path-dependent and the value of the steady state opinion depends on the activation order of agents.

When \( \delta = \sigma = 0.5 \), Figure 30 shows one realization of the opinion dynamics with the steady state opinion equal to \( [0.119, 0.119, 0.039, 0.039, 0.802, 0.411, 0.915, 0.915, 0.915, 0.915] \).
Figure 30: Opinion dynamics for two communitarian groups of Example 13 with $\delta = \sigma = 0.5$.

5 Concluding remarks

This paper proposes two models of continuous opinion dynamics in undirected networks, by introducing the heterogeneity either into nodes or into links in the sense of conformity and anti-conformity behavior. We propose an appropriate updating rule of continuous opinions for anti-conformity behavior, defined according to the repelling function, which gives the shift of the opinion based on the current opinion and the reference opinion for an agent. Both synchronous and asynchronous opinion updates are studied in these two models. In synchronous models, all agents are assumed to have the opinion updates simultaneously, while in asynchronous models, an agent is chosen at random to be active following the same updating rule as in synchronous models, and the opinions of other inactive agents stay unchanged.

In the first part of the paper, the model of opinion dynamics is studied with both conformist and anti-conformist agents. Conformist agents update opinions according to the DeGroot rule with equal weights on her neighbors, while anti-conformist agents deviate from the average opinions of her neighbors. For any connected network, consensus will never be reached as long as the set of anti-conformist agents is nonempty in both synchronous and asynchronous models. Instead, opinions of agents oscillate or converge to a disagreement, which is more common in real life. When one anti-conformist agent is connected to a set of connected conformist agents, then opinions do not converge regardless of the value of initial opinion. Instead, we see from the simulations that opinions are periodically oscillating between 0 and 1 for the anti-conformist agent, and between $\epsilon'$ and $1 - \epsilon'$ (where $\epsilon' < \epsilon$) for the conformist agents symmetrically in synchronous models, in the sense that the increasing process from 0 (resp., $\epsilon'$) to 1 (resp., $1 - \epsilon'$) of opinions is symmetric to the decreasing process from 1 (resp., $1 - \epsilon'$) to 0 (resp., $\epsilon'$). Although the non-convergence and oscillations also hold for asynchronous models, the periodicity...
and symmetry disappear, since the dynamics of the asynchronous updates become path-dependent on the activation order of the agents. For the case of anti-conformist agents being extremist and conformist agents being moderate, under mild conditions on the initial opinions and the number of neighbors in each group (such that the influence from two extreme groups are balanced), the ordering consistency and opinion convergence hold, so anti-conformist agents remain extremist and conformist agents continue to be moderate. The exact value of the steady-state opinion is given when \( \sigma = 1 \), i.e., when the repelling interval is maximal. This result holds for both synchronous and asynchronous opinion updates.

In the second part of the paper, the model of opinion dynamics is studied over signed graphs with both positive and negative influence. We show that for two communitarian groups (i.e., structurally balanced graphs) with disjoint initial opinions, if each sub-network (i.e., each communitarian group) is connected, then the ordering consistency and opinion convergence holds, and opinions of agents from the same group form a consensus, for both synchronous and asynchronous opinion updates. In addition, the steady-state opinion is characterized for the case that two groups are connected. However, when considering more than two communitarian groups and synchronous updating, all the properties such as ordering consistency, opinions convergence may fail, and oscillation can sometimes happen, depending on the values of \( \delta \) and \( \sigma \). In the asynchronous updating model, ordering consistency may fail, but opinions always converge, regardless of the values of initial opinions and the values of \( \delta \) and \( \sigma \). But the steady-state opinion is not pre-determined, since the dynamics are also path-dependent and the activation order of agents matters.

If we exclude the case when the oscillations are only caused by the synchronization like in the example of two anti-conformist agents with the same opinions, in general, the asynchronization of opinions updates do not change the convergence behavior of the opinions but the speed of the dynamics. This is also in accordance with the results of Ramazi et al. (2016), emphasizing that the asynchrony does not lead to cycles or non-convergence. In other words, the asynchrony preserves the convergence property of the corresponding synchronous models, and sometimes even turns the oscillation behavior shown in the synchronous model into convergence.

The updating rule of continuous opinions proposed in this paper is flexible and appropriate for modeling anti-conformity behavior for the following reasons. First of all, the driving force of anti-conformity urges agents (anti-conformist agents or agents with negative influences) to repel from the reference opinion, which can take various forms depending on the context. For example, it is the average opinion of agents in one’s neighborhood in the CODA-node model and the average opinion of one’s enemies in the CODA-link model. It can also be the average opinion of agents from the entire society in the fashion context since everyone including the anti-conformist agent can easily obtain relevant information about fashion trend due to the popularity of the Internet. When modeling from a game-theoretic point of view, it can also take value of the average opinion of agents from a certain group. Secondly, the results presented in the current paper apply to different forms of the repelling functions such as, but not limited to, convex, concave or linear functions as shown in the examples. Thirdly, the deviation of the opinion for an agent is increasing as the reference opinion becomes closer to her current
opinion, it implies the idea of always distancing the others. This sometimes causes oscillations such as in the very simple case of one anti-conformist agent linked to one (or a set of connected) conformist agent(s) in the CODA-node model, the anti-conformist agent playing the role of a leader (e.g., of a certain fashion), always followed by the conformist agent(s). Recall that opinions are defined in the close interval \([0, 1]\) where the two boundaries are referred to as the two extreme opinions. Once the anti-conformist agent reached some extreme opinion, she would lead the others back to some mild opinions, which can explain well the fashion fluctuations. Opinions can also reach equilibria. For example, in the case of two groups of anti-conformist agents holding relatively extreme opinions and conformist agents holding relatively moderate opinions, opinions converge when the influences from two different groups of anti-conformist agents are balanced. Moreover, as each anti-conformist agent has her own repelling interval \([-\sigma_i, \sigma_i]\) within which she is influenced, her new neighborhood at \(t + 1\) would be \(\{j \in N : |x_j(t + 1) - x_i(t + 1)| < \sigma_i\}\). Thus the CODA-node model can also be seen as a coevolution model of networks and opinions.

Even though models of opinion dynamics with anti-conformity behavior is drawing attention in recent years, mainly on binary opinions, the study of continuous opinions with anti-conformity behavior still requires to be developed. To the best of the author’s knowledge, there are two related works on continuous opinion dynamics. As mentioned in Section 1, Buechel et al. (2015) assumed that each agent is assigned with a level of conformity, measured by a parameter taking value in \([-1, 1]\), where -1 and 1 refer to full anti-conformity and full conformity, respectively. By contrast, in the current paper, we suppose no continuum in between conformity and anti-conformity. Another work is given by Altafini in several papers, who focused on the opinion dynamics with negative influences. As shown by Example 5, this model is not appropriate for describing anti-conformity behavior. Moreover, the study of opinion dynamics with anti-conformity behavior also requires the empirical and experimental support. Future studies include testing the new updating rule with some actual data, introducing strategic network formation into the model, and considering directed networks instead of a fixed undirected network supposed in the current paper.
Appendices

A Non-negative matrices

By convention, transposition is denoted by ‘. Let \( T \) be a nonnegative \( n \times n \) matrix and \( N := \{1, 2, \ldots, n\} \). To any nonnegative matrix \( T = [T_{ij}] \) we associate a directed graph \( \Gamma \) with set of nodes \( N \), and the set of arcs \( \{(i, j)|i, j \in N, t_{ij} > 0\} \). A walk of length \( k \) from node \( i \) to node \( j \) is a sequence of nodes \( i = i_0, i_1, \ldots, i_k = j \) such that \((i_{l-1}, i_l)\) is an arc in \( \Gamma \) for \( l = 1, \ldots, k \). \( i \rightarrow j \) refers that there is a walk from \( i \) to \( j \). A cycle around \( i \) is defined as a walk from \( i \) to \( i \) which does not pass through \( i \) between the starting and the ending points.

A class is a set of nodes \( C \) such that either \( C \) is a singleton or \( i \leftrightarrow j \) for every distinct \( i, j \in C \), and any \( C' \subset C \) does not full the latter property. A class is essential if no arc is going out of it, otherwise it is inessential.

The canonical form of a matrix \( T \) with \( q \) essential classes and \( w \) inessential classes is

\[
T = \begin{bmatrix}
T_1 & 0 & \cdots & 0 \\
0 & T_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & 0 \\
0 & \cdots & 0 & T_q \\
R & 0 & \cdots & 0 \\
\end{bmatrix}
\]

with

\[
Q = \begin{bmatrix}
Q_1 & 0 & \cdots & 0 \\
& Q_2 & \ddots & \vdots \\
& & \ddots & 0 \\
& S & \cdots & Q_w \\
\end{bmatrix}
\]

where elements in \( N \) have been ordered so that essential classes come first (in any order), then inessential classes, so that if for any \( i \) and \( j \) in two distinct inessential classes, \( i \) is ranked before \( j \), we have \( i \not\rightarrow j \).

Fact 4. \( \lim_{k \to \infty} Q^k = 0 \).

Lemma 1. Let \( A \) be a finite \( n \times n \) matrix such that \( \lim_{k \to \infty} A^k = 0 \). Then \( [I - A]^{-1} \) exists and \( [I - A]^{-1} = \sum_{k=0}^{\infty} A^k \) with \( A^0 = I \).
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