Leverage cycles when banks have a choice

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Abstract

Leveraged investing can aggravate crises when dropping asset prices force sales in falling markets. Leverage control policies introduced to counteract this could lead to agents targeting the maximal leverage deemed ‘prudent’ at all times. Aymanns and Farmer (2015) show how, for a Value at Risk type control policy, this ‘leverage targeting’ can not only lead to aggravated crises, but can also trigger the crisis to begin with and cause the boom that follows it. In modelling a whole cycle, however, the assumptions that investors base their demand for risky assets only on their leverage target becomes more dubious. In this paper, we allow our agents to repeatedly choose between a leverage targeting and a (leverage constrained) expected utility optimisation strategy. We find that both strategies persist, and in fact are equally prevalent on average. The endogenous cycles that Aymanns and Farmer (2015) found persist, although their amplitude is reduced. Agents will thus regularly choose to invest up to their leverage constraint, without further consideration of the state of the market, even though this destabilises the market. We find that allowing short-selling for agents using the optimisation strategy does stabilise the market.

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1 Introduction

The use of borrowed money to invest, a practice known as leverage, has been identified as one of the aggravators of the great financial crisis of 2008. But even before this, in the seminal work by Geanakoplos (2003), it was pointed out that leverage can exacerbate crises by forcing sales into falling markets. Aymanns and Farmer (2015) on the other hand shows that leverage cannot only exacerbate an existing crisis, but can actually be the cause of a full cycle.

In a different strand of literature focused on systemic stress testing in financial systems (e.g. Cont and Schaaning (2017), Bookstaber et al. (2018)), it has been identified that the tendency of banks to target a specific leverage aggravates crises, even if any expectations or risk estimates are ignored. When asset prices fall, leverage increases automatically. To get back down to the original leverage, assets have to be sold, leading to further price falls and further increasing leverage. The wish to target a specific leverage is not necessarily voluntary, under Basel II regulation for example banks are faced with explicit constraints on their leverage.

While these arguments only show how leverage and leverage targeting can aggravate crises, Aymanns et al. (2016) show how these practices can also cause the next bubble and following crash to form, even without any exogenous shocks. Their banks have a more sophisticated leverage target than most of the systemic stress testing literature, namely one that depends on recent risk estimation. They do not, however, form any view on the direction in which they believe the price of a risky asset will move. While this is a defensible position in times of crisis, when banks will focus on preventing the breach of constraints (Bookstaber et al. (2018)), it is not necessarily intuitive during other parts of the business cycle. Why would one blindly buy as many risky assets as one is allowed to, without considering whether this will be a good investment based on funding costs, dividends and expected price-changes? Data on leverage and perceived risk supports a less clear-cut correlation between these two variables.

In this paper, we allow the banks in our model to choose between the leverage targeting strategy from Aymanns et al. (2016) and a strategy based on constrained expected utility maximisation which we’ll call leverage optimising. They reassess what strategy they would like to use in every time step, using a Heuristic Switching Model to make their choice Brock and Hommes (1997). In doing so we investigate if and when the leverage targeting strategy is attractive to banks, and we assess to what extent the broad destabilising effect of leverage is specific to models where leverage is the sole focus of banks. Our model has two types of agents: banks and fundamentalists. They trade in a risky asset and a risk-free asset. As opposed to Aymanns et al. (2016), our fundamentalist uses the familiar myopic mean variance wealth maximisation. This means it sells (goes short) when the asset is valued above its fundamental value, and buys when it is below this value. Our banks are not allowed to go short, since there
is no clear way to implement going short in the face of a leverage target.

First of all, we study two simplified models where either all banks use leverage targeting or all banks use leverage optimising. When all banks use leverage targeting we are in almost exactly the situation of Aymanns et al. (2016), with the difference of a slightly different fundamentalist, and we find the chaotic leverage cycles or convergence to a fixed point. The chaotic cycles work as follows. After a period of relative stability, the volatility estimate is low. The leverage target is thus high, and the leverage targeter will have a high demand. This drives the price of the asset up. After a while, due to a memory parameter $\delta$, the bank’s volatility estimate starts picking up. Thus the leverage target starts dropping, and the leverage targeter will have a lower demand. This drives the price of the asset down again, and we enter a period of relative stability until the crisis starts to fade from memory and the volatility estimate drops again. Under leverage optimising for all banks however we find convergence to the fundamental fixed point in all cases. When a crisis occurs that brings the price of the risky asset close to (but above) the fundamental value, both the bank and the fundamentalist will have the expectation that the price will remain close to the fundamental value, and as such the fundamentalist will be short the asset and the bank will be slightly long the asset until the price stabilises at the fundamental value.

When we simulate our full model we find that leverage and its constraints still cause both bubbles and crashes without any exogenous triggers. During a boom, all banks start acting like leverage targeters since this is the most profitable strategy. This is because during a boom, only the leverage targeting strategy is long the asset. Eventually we know the leverage targeting strategy leads to a crash. During the crash, it is best to use the leverage optimising strategy because they have a lower demand. Thus everyone switches rapidly to the leverage optimising strategy. As a result, the price drops even more quickly than in the case of leverage targeting. We then enter a period of relative stability. The leverage targeter has an increasing demand due to a monotonic decreasing of the volatility estimate, but the leverage optimiser has a demand that decreasing at roughly the same rate. The banks are split about evenly across the two strategies. This continues until the leverage optimiser hits their short-selling limit. Then the monotonic increasing demand of the leverage targeter leads to a new boom, and the cycle is complete. The cycles, however, have a smaller amplitude than in the case of leverage targeting only. This is because the leverage optimisers are still present at the beginning of a boom, and have a lower demand, meaning that the total demand by banks is a bit lower.

We study the effects of four parameters on the dynamics of our model. Parameter $\beta$ determines the intensity of choice between the two strategies. The higher it is, the more banks will choose the more profitable strategy. We find that increasing $\beta$ increases the amplitude of cycles, due to a more pronounced herding effect at the top and bottom of a cycle. Parameter $\delta$ is the aforemen-
tioned memory parameter in the volatility estimate. The amplitude of cycles increases in $\delta$. This has to do with the fact that an increased $\delta$ initially leads to smaller amplitude cycles, since the leverage optimiser will have a higher demand during the crash when $\delta$ is larger, since the boom has a less prominent role in the volatility estimate. After a number of small amplitude cycles the volatility estimate can reach exceptionally low levels, leading to an extra large boom and crash. Parameter $f$ determines what fraction of our agents are fundamentalists. If $f$ is very large, we find convergence to a fixed point. The amplitude of the cycles increases as $f$ decreases, since prices change more based on changes in the banks demand if the fraction of fundamentalists is smaller. The final parameter we study is $\sigma^0$, the floor of the volatility estimate. For large $\sigma^0$, we find convergence to a fixed point. The leverage targeter demand is capped at such a low demand that their destabilising effect is curbed completely. The amplitude of cycles increases as $\sigma^0$ decreases. When $\sigma^0$ is smaller, the leverage targeter can buy more during the boom, which results in larger price spikes.

Allowing those banks who follow the optimising strategy to short-sell removes the bubbles and crashes. This is because in the original model, the end of the period of stability and beginning of a new boom occurs when the leverage optimiser reaches their short-selling limit. When the optimiser can short-sell, we see their decreasing demand compensate for the increasing demand of the leverage targeter. As a result the net demand of the banks is about 0, and the fundamentalist determines that the price will reach the fundamental value and stays there.

This paper is structured as follows. We begin with a literature review, and then present our model. In Section 4 we analyse some special cases of the model, to aid us in Section 5 where we consider the full model dynamics. In Section 6 we focus specifically on the competition between the two strategies. Section 7 discusses the effect of a policy permitting short selling on the dynamics, and Section 8 concludes.

## 2 Literature Review

Danielsson and Zigrand (2003) and Danielsson et al. (2004) study the impact of a Value at Risk constraint. Their agents invest their wealth (without leverage) into cash and $N$ different risky assets. They choose their portfolio weights through VaR-constrained expected utility maximisation. The first paper considers a single period general equilibrium model, the second a multi-period learning framework where agents revise their beliefs on the expected returns and (co)variances of the risky assets every time step. The true pay-offs of the risky asset in their model are exogenously generated, so independent of the demands of the agents. A few conclusions are drawn about the macro-prudential consequences of the VaR constraint, such as lower average prices and higher volatility.

Geanakoplos introduced the name ‘leverage cycle’. In a number of papers, com-
ing together in Geanakoplos (2010), he lays out his theory. His first contribution is to show that the equilibrium of supply and demand can determine not only the interest rate on loans, but also the leverage (or its inverse, the haircut on a loan). He then uses this theory of equilibrium leverage to explain the anatomy of a big crash following ‘scary bad news’: bad news that not only lowers expectations but also increases uncertainty and disagreement. Geanakoplos shows that when scary bad news hits, the losses in asset value from lowered expectations are severely aggravated through tightening leverage constraints, forcing even optimists to sell. He does this in a three period binary tree general equilibrium model. Importantly, Geanakoplos agents take rational decisions given their expectations.

Thurner et al. (2012) and Poledna et al. (2014) develop an agent based model with different types of agents. In the latter, more extensive, one there are four: fund managers, fund investors, banks and noise traders. The fund managers are the key agents for our purposes: their demand for the risky asset is based on deviations from an exogenous fundamental value, but they are also faced with a leverage constraint. This constraint is passed on through the banks who lend but abide by regulation on their credit exposure. Their model produces repeated leverage cycles, with the downturn always caused by a random shock to the noise trader demand. There is one risky and one risk free asset.

Aymanns and Farmer (2015) and Aymanns et al. (2016) develop a simpler model which is analytically tractable. They produce leverage cycles, both the way up and the way down, even in the deterministic limit. These are mainly driven by the method of risk estimation used by their main agents (banks). A second agent, a kind of market maker, has a stabilising effect on the market. Their model is similar to the one in Danielsson et al. (2004), except that agents are now allowed to use leverage. Banks first determine how much leverage the VaR rule allows them to take on, given their estimation of the volatility of the risky asset. This translates directly into a dollar value of assets, and due to an exogenous weight on bonds, into a demand for the asset. The leverage cycles are driven by the method of risk estimation and the leverage taken on as a result. In short, when the volatility estimate is low banks take on a lot of leverage, which drives up the price of the risky asset. This makes the volatility estimate go up, so that banks have to delever, which involves selling assets and thus reducing the price. The dropping price increases the volatility estimate even more. When the banks are at minimal leverage, they are not buying/selling much, and the price stays more stable. Slowly the boom and bust fade from memory, the volatility estimate drops and banks take on more leverage again.

Adrian and Shin (2010) present data suggesting that for investment banks, leverage is pro-cyclical: banks will increase their leverage when asset prices are rising and reduce it in bad times. In particular, they show that these investors will respond to changes in the value of their assets by changing their leverage rather than changing their equity. These findings form key empirical backing
for the assumptions in the existing literature. The empirical part of Adrian and Shin (2013) consists of further analysis of data similar to that in Adrian and Shin (2010), importantly now including data collected during the financial crisis. They focus on the explanatory role played by firms’ estimation of risk as measured through the Value at Risk they report. They find evidence that banks use a Value at Risk rule when determining their leverage, and further evidence that they do this through changes on the asset side rather than by altering their equity. They do not, however, find the -1 correlation between changes in leverage and changes in perceived risk that is predicted by the VaR rule. The correlation is negative, but by far not as strong as would be expected, at about -0.5.

We see this somewhat weak correlation between perceived risk and leverage as evidence against the behavioural rule governing leverage in Aymanns and Farmer (2015). Their agents will always take on the full leverage allowed given their perceived risk, and as a result their simulations give for a perfect −1 correlation between perceived risk and leverage. In Aymanns et al. (2016) the strength of the correlation is only reduced through an exogenous parameter controlling the speed of adjustment by the banks. We do not believe this is a sufficient explanation for the limited correlation. Especially outside of times of stress, banks may not always want to take on the full risk they are allowed to. This could, for example, be because they expect a downturn in the stock market, even if their volatility estimate is not yet very high. In our model we will allow for this possibility, thereby representing the more complicated relationship between perceived risk and leverage that we see in the data.

The agents in Aymanns et al. (2016) base their entire investment decision on the leverage they would like to achieve. We will refer to this practice as ‘leverage targeting’, since the leverage is consciously chosen rather than mostly a side-effect of other considerations. Modelling agents this way is by no means unique to the papers by Aymanns, Farmer et al, and is often justified by referring to Adrian and Shin (2010). Sometimes the agents will have a constant leverage target in mind, independent of the conditions they are faced with (e.g. Greenwood et al. (2015), Duarte and Eisenbach (2018)). In other papers the agents allow for some fluctuation around their leverage target, but will take action to return to their leverage target when some buffer value is reached (e.g. Bookstaber et al. (2018), Cont and Schaanning (2017)). What these papers tend to have in common is that they deal with crisis situations, where banks are barely attempting to make decisions to chase profits and instead are simply trying not to breach any constraints. Aymanns et al. (2016) explicitly does not solely consider crisis situations, but rather both the boom and the bust phase of a financial cycle. They also have the most dynamic leverage target, which constantly adapts to the state of the financial system.
3 Model

3.1 Description

We set up a model similar to that of Aymanns et al. (2016), with the main exception that the ‘bank’ agents can choose between two strategies using heuristic switching. One strategy is almost exactly that of the original model (‘leverage targeting’), the other allows for a constrained expected utility maximisation to decide whether the bank actually wants to take on the full risk it is allowed to (‘leverage optimising’). The noise trader in their model is replaced by a ‘fundamentalist trader’ which simplifies the model and fits with more traditional asset pricing models.

Consider an asset pricing model with one risky asset and one risk free asset. The risk free asset (cash) pays a fixed return \( R = r + 1 \). For the sake of simplicity we say that the interest agents pay on any loans, as well as the interest paid per unit value of stock loans is equal to \( r \). The risky asset has price \( p(t) \) per share at time \( t \), and pays a dividend \( y \).\(^1\)

We have two types of agents: banks and fundamentalist traders. A bank is faced with a leverage constraint and is not allowed to short sell. Their leverage constraint is based on their estimate of the variance of the risky asset in each time step. They choose between two strategies: either they always adapt their balance sheet to exactly reach the maximal allowed leverage (leverage targeting), or they form an expectation of the future value of the risky asset to decide whether they would actually prefer taking on less risk than allowed (leverage optimising). In either case they are faced with the leverage constraint, but in one case they blindly follow it while in the other case they also assess how attractive the risky asset is to them.

The choice between the two strategies is made through the heuristic switching model (Brock and Hommes (1997)). Every time step, we calculate the profits one would have made if one had chosen that strategy, and then the strategy with the most profits attracts the most followers. The fraction of banks who follow the targeting strategy instead of the optimising strategy is thus endogenous. The fraction of banks versus fundamentalists is exogenous, representing the size of the banking sector versus the rest of the economy.

The fundamentalists behave as if they were in a homogeneous rational world. They have a constant belief on the fundamental value and variance of the risky asset, and are allowed to short-sell. They do not leverage their investments. These traders represent “the rest of the economy”.

\(^1\)For most of the paper we will consider the deterministic dynamics, where the dividends are constant \( y \). In the appendix we have included a numerical simulation where \( y_t = \bar{y} + \epsilon_t \) where \( \epsilon_t \) is IID with a uniform distribution on a small interval \([-\epsilon, +\epsilon]\) as in (Brock and Hommes, 1998).
The price of the risky asset is determined through equating supply and demand, assuming an outside supply of 0.

A derivation of the model equations follows. The details can be found in the appendix.

### 3.2 Fundamentalist

The demand of the fundamentalist trader closely follows Brock and Hommes (1998). We assume that the trader is a myopic mean variance wealth maximiser with risk aversion parameter $\tilde{\alpha} > 0$. They do not borrow cash, but may borrow stocks to engage in short selling. We assume that $\mathbb{E}_t p_{t+1}$ and $\text{Var}_t p_{t+1}$ are both constant in time for this trader, which is the feature that earns them their fundamentalist title. Their expectation is always that the price of the risky asset will return to its fundamental value $\tilde{p} = \frac{\bar{y}}{1-R}$, and they always think the variance of the risky asset is $\sigma^2$. By letting $a = \tilde{\alpha} \sigma^2$ we then find the demand of the fundamentalist trader to be:

$$z_t = \frac{\tilde{p} + \bar{y} - Rp_t}{a}$$

### 3.3 Bank

Banks are investors who are allowed to borrow cash to leverage their investments, but are not allowed to short sell. They are subject to a leverage constraint, which arises from a Value at Risk policy which is so that portfolio losses have at most probability $c$ to exceed some dollar value. Such a constraint looks like:

$$\frac{\text{Assets}}{\text{Equity}} \leq \frac{\alpha}{\sigma_t}$$

Where $\frac{\text{Assets}}{\text{Equity}}$ is equal to the leverage, $\alpha > 0$ is a parameter of the model derived from the aforementioned probability $c$, and $\sigma_t$ is the bank’s current estimate of volatility of the risky asset. They form this estimate through an exponential moving average:

$$\sigma_t^2 = (1 - \delta)\sigma_{t-1}^2 + \delta \log \left( \frac{p_t}{p_{t-1}} \right)^2$$

where $0 < \delta < 1$ is a parameter.

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2 We do not allow the banks to short-sell because this is very hard to rhyme with leverage targeting. The leverage target only tells you how big of a position you want to take, it cannot tell you whether this position should be positive or negative. To make the strategy switching for the banks a fair choice, where they are subject to the same rules in both cases, we thus forbid short selling altogether. This is by no means exceptional in the literature. In Section 7 we remove this constraint for the leverage optimising strategy.
We then assume the bank sector to have constant equity \( E \), which is as in Aymanns and Farmer (2015) and supported by the findings in Adrian and Shin (2013).

Given this setting, the bank can now choose between the Leverage Targeter and the Leverage Optimiser strategy.

### 3.3.1 Leverage targeter

The leverage targeter will always make sure that they operate at the maximal allowed leverage. As in Aymanns et al. (2016), we introduce a volatility offset parameter \( \sigma^0 > 0 \), which places an effective floor on the volatility estimate the bank uses. This reflects that even in stable times, it is common knowledge that some noise is to be expected. It also enforces a maximal demand available to the leverage targeter.

Using (2) it follows that the demand of the leverage targeter is given by:

\[
z_{1,t} = \frac{\alpha E}{p_t \sqrt{\sigma_t^2 + \sigma^0}}
\]  

### 3.3.2 Leverage optimiser

The leverage optimiser does not necessarily operate at maximal allowed leverage. They form an expectation of the price of the risky asset in the next time step, and then perform myopic mean variance wealth maximisation, with risk aversion parameter \( 1 \). The variance they use in this maximisation is equal to the estimated volatility (3). The expectation in this maximisation is formed as a moving average as follows:

\[
\mu_t := E_t p_{t+1} = (1 - \delta) \mu_{t-1} + \delta p_{t-1}
\]

with the same parameter \( \delta \) as in (3). This is similar to the expectation formation in Danielsson et al. (2004).

While the leverage optimiser forms its ideal demand through this myopic mean variance maximisation, they also are bound by both the aforementioned leverage constraint and the shortselling constraint. The demand of the leverage optimiser is therefore constrained by 0 and the demand of the leverage targeter. Their actual demand for the risky asset is thus:

\[
z_{2,t} = \min \left( \max \left(0, \frac{\mu_t + \bar{y} - R p_t}{a(\sigma_t^2 + \sigma^0)} \right), \frac{\alpha E}{p_t \sqrt{\sigma_t^2 + \sigma^0}} \right)
\]

### 3.3.3 Switching

The banks choose between the targeter and optimiser strategies through the heuristic switching model from Brock and Hommes (1997). First, the profits
that a bank using strategy $i$ would have made in the past time step are calculated, and denoted by $\pi_{i,t}$. We then assume there is a continuum of banks, and that the fraction of those banks who decides to use strategy $i$ at time $t$ is equal to:

$$n_{i,t} = \frac{e^{\beta \pi_{i,t}}}{e^{\beta \pi_{1,t}} + e^{\beta \pi_{2,t}}}$$  \hspace{1cm} (7)

### 3.4 Price

The price of the risky asset is determined by equating supply and demand. The demand comes from the fundamentalist and the banks, and the supply is fixed at 0. We have a fixed fraction of fundamentalist traders, $f$, so the actual fraction of traders which are both bank and use leverage targeting is $(1-f)n_{1,t-1}$ and similarly for leverage optimisation.\(^3\)

The supply/demand equation is thus:

$$f z_{f,t} + (1-f)n_{1,t-1} z_{1,t} + (1-f)n_{2,t-1} z_{2,t} = 0$$

Since the demand function of the leverage optimiser (6) consists of three parts, we solve for the resulting equilibrium price in each of them. $p_{0,t}^*$ gives the price if the leverage optimiser has 0 demand, $p_{\mu,t}^*$ gives the price if the demand is based on the expectation $\mu$, and $p_{C,t}^*$ the price if the leverage optimiser is operating at its leverage constraint.

$$p_{0,t}^* = \frac{\hat{p}}{2} + \sqrt{\frac{\hat{p}^2}{4} + a\alpha n_{1,t-1} \frac{E}{\sigma^2 + \sigma^0} \frac{(1-f)}{Rf}}$$  \hspace{1cm} (8)

$$p_{\mu,t}^* = \sqrt{(\hat{p} + (1-f)n_{2,t-1} \frac{(\mu + \bar{y})}{\sigma^2 + \sigma^0})^2 + 4(R + \frac{R(1-f)n_{2,t-1} a\alpha n_{1,t-1} E}{\sigma^2 + \sigma^0} \frac{(1-f)}{f})} - 2(R + \frac{(1-f) R n_{2,t-1}}{f(\sigma^2 + \sigma^0)})$$  \hspace{1cm} (9)

$$p_{C,t}^* = \frac{\hat{p}}{2} + \sqrt{\frac{\hat{p}^2}{4} + a\alpha \frac{E}{\sigma^2 + \sigma^0} \frac{(1-f)}{Rf}}$$  \hspace{1cm} (10)

And since (6) is monotonic in $p$ the equilibrium price is unique and given by:

$$p_t = \min(\max(p_{0,t}^*, p_{\mu,t}^*), p_{C,t}^*)$$  \hspace{1cm} (11)

\(^3\)We need $n_{1,t-1}$ here because we need to use a fraction based on already realised profits. This requires the demand from two time steps ago, which last time step resulted in profits/losses based on the realised price at $t - 1$. 

10
3.5 Model as a dynamical system

We can write the model as a first order five dimensional dynamical system as follows:

3.5.1 State variables and system

The model consists of five state variables and hence five equations govern the dynamics:

\[ p_t'(p) = p_{t-1} \]

\[ \sigma_t^2(p', \sigma^2, p) = (1 - \delta)\sigma_{t-1}^2 + \delta \log \left( \frac{p_{t-1}}{p_t} \right)^2 \]

\[ \mu_t(\mu, p) = (1 - \delta)\mu_{t-1} + \delta p_{t-1} \]

\[ p_t(p', \sigma^2, \mu, p, n_1) = \min(\max(p_{0,t}, p_{\mu,t}^{\ast}), p_{C,t}^{\ast}) \]

\[ n_{1,t}(p', \sigma^2, \mu, p, n_1) = \frac{e^{\beta \pi_{1,t}}}{e^{\beta \pi_{1,t}} + e^{\beta \pi_{2,t}}} \]

To simplify these expressions we use the following auxiliary variables:

\[ \pi_{1,t} = \frac{\alpha E}{p_{t-1} \sqrt{\sigma_{t-1}^2 + \sigma_0^2}} (p_t + \bar{y} - R p_{t-1}) \]

\[ \pi_{2,t} = \min \left( \max \left( 0, \frac{\mu_{t-1} + \bar{y} - R p_{t-1}}{\alpha (\sigma_{t-1}^2 + \sigma_0^2)} \right), \frac{\alpha E}{p_{t-1} \sqrt{\sigma_{t-1}^2 + \sigma_0^2}} (p_t + \bar{y} - R p_{t-1}) \right) \]

\[ n_{2,t} = 1 - n_{1,t} \]

and \( p_{0,t}^{\ast}, p_{\mu,t}^{\ast}, p_{C,t}^{\ast} \) defined in (28), (29) and (30).

We keep track of the weighted average leverage of the banks (since this is a quantity we can compare to published data). Since the banks who use strategy i hold \( z_i(t) \) units of the risky asset, and the leverage is equal to the ratio of risky assets to equity, we can write the bank leverage as:

\[ \lambda_t = \frac{n_{1,t-1} z_{1,t} p_t + n_{2,t-1} z_{2,t} p_t}{E} \]
3.6 Parameters

The parameters of the model are: \( \bar{y}, \beta, \delta, R, \alpha, E, a, f, \sigma_0 \). \( \bar{y} \) determines the mean dividends, while \( R \) determines the risk free rate. \( \delta \) and \( \sigma_0 \) play a role in the volatility estimate as the memory parameter and the floor of the estimate respectively. The leverage target is governed by \( \alpha \), the risk level of the leverage targeter, and \( E \), the banks’ equity. Finally \( f \) determines the fraction of fundamentalists, and \( \beta \) the intensity of choice of the heuristic switching model governing the strategy being used by the banks. In our analysis we focus mainly on the following parameters: \( f, \delta, \beta \) and \( \sigma_0 \). Table 1 shows all the parameters of the model and their values and origins.

Table 1: Default values for the parameters

<table>
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<th>Parameter</th>
<th>Description</th>
<th>Default</th>
<th>Reason</th>
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| \( \bar{y} \) | Mean dividends | 0.02 | Mean S&P 500 dividends (adjusted for inflation) for the past 10 years are 2%.
| \( R \) | Risk free rate | 1.006 | Mean effective federal funds rate over the past ten years.
| \( \alpha \) | Risk level leverage targeting | 0.75 | Chosen to target a reasonable range of values for leverage.
| \( E \) | Bank’s equity | 2.27 | Aymanns et al. (2016).
| \( a \) | Risk aversion | 1 | Normalised.
| \( \sigma_0 \) | Volatility offset | \(10^{-6}\) | Aymanns et al. (2016).
| \( f \) | Fraction fundamentalists | 0.5 | Varied in the analysis in the range [0, 1]. Default value arbitrarily at the midpoint.
| \( \beta \) | Intensity of choice strategy | 1 | Estimated by Hommes et al. (2017).
| \( \delta \) | Memory parameter | 0.2 | Default value from chosen to replicate empirical frequency of crises.

4 Illustrative special cases

Before analysing the whole model, we will go through three special cases to illustrate some of the model dynamics.

4.1 Only the fundamentalists

If \( f = 1 \) so that we only have fundamentalist investors, we always find convergence to the fundamental steady state, which is:

\[
(p^*, \bar{p}^*, \hat{p}, \hat{p}, 0) = \left( \frac{\bar{y}}{R - 1}, \frac{\bar{y}}{R - 1}, \frac{\bar{y}}{R - 1}, 0 \right)
\]

This is because the fundamentalist will be short the asset when it is valued above the fundamental value, and long when it is valued below the fundamental value. As a result it is only possible to clear the market with only fundamentalists present if the asset is valued exactly at the fundamental value.

4.2 All banks leverage targeters

In this section we focus on the case where all banks act as leverage targeters. If we fix \( n_1(t) = 1 \), all banks are leverage targeters and on top of that we have \( f \)
fundamentalist investors. This leads to chaotic cycles or convergence to a fixed point. This makes sense since our leverage targeter strategy is very similar to the bank in Aymanns and Farmer (2015).

The chaotic cycles illustrated in Figure 1 work as follows. After a period of relative stability, the volatility estimate is low. The leverage target is thus high, and the leverage targeter will have a high demand. This drives the price of the asset up. This can go on for a while before the volatility estimate really starts picking up, due to the memory parameter $\delta$ in the volatility estimate. This means that as the information on the new boom is coming in, the information about the previous crisis is still slowly being forgotten, which can initially roughly cancel each other out. When the price has been building up for a while, however, there comes a point where this starts having enough of an impact on the volatility estimate that the estimate increases. We have now reached the maximum price. The leverage targeter is now faced with a lower leverage target, and thus a lower demand. This has a negative impact on the price, and it starts dropping. Meanwhile the volatility estimate increases further, since the pre-boom period is fading from memory and new information coming through is of volatile prices. Thus the targeter has even lower demand, the price drops fur-
ther, and the volatility estimate increases even more. Eventually the volatility
estimate is very high and the leverage targeter has gotten rid of almost all their
assets. We then enter a period of relative stability, where the price is increasing
but at a much lower speed than we saw the decrease happen during the crash.
This continues until the memory of the crisis starts to fade and the cycle starts
over.

In the unique fixed point, the leverage targeters operate at their demand ceiling
as imposed by $\sigma^0$. We can find it as follows:

$$p^{1,*} = p^*$$

$$(1 - \delta)\sigma^{2,*} + \delta \log(\frac{p^*}{p^*}) = \sigma^{2,*}$$

$$(1 - \delta)\sigma^{2,*} + \delta 0 = \sigma^{2,*}$$

$$(1 - \delta)\sigma^{2,*} = \sigma^{2,*}$$

So that either $\delta = 0$ or $\sigma^{2,*} = 0$.

We can then find $p^*$ by equating supply and demand:

$$\frac{\tilde{p} + \tilde{y} - Rp^*}{a} + (1 - f) \frac{\alpha E}{\sqrt{\sigma^0}p^*} = 0$$

$$p^* = \frac{\tilde{p}}{2} + \sqrt{\frac{\tilde{p}^2}{4} + \frac{a\alpha E(1 - f)}{fR\sqrt{\sigma^0}}}$$

Note that this price is higher than or equal to the fundamental price $\tilde{p}$ (see
Figure 2), and tends to $\tilde{p}$ as, for example, $E$ tends to 0 or $f$ tends to 1. For the
standard parameters however this price is much higher than $\tilde{p}$ since $\sigma^0$ is chosen
close to 0. In fact, in the fixed point the leverage targeter is precisely operating
at their demand ceiling, which is imposed by the introduction of $\sigma^0$.

Stability analysis shows that the eigenvalues of the Jacobian are 0, 0, $1 - \delta$.
Thus the fixed point is locally stable for all $\delta < 1$.

When $\sigma^0$ decreases, the ceiling of the targeter demand is higher and the equi-
librium price is higher. The relative impact of changes in $\sigma^2$ also increases. We
thus see, from Figure 3, that when $\sigma^0$ becomes small enough, cycles crash long
before the equilibrium price is reached, and we thus see the emergence of chaotic
behaviour (as confirmed by analysis of the Lyapunov exponent). At the same
time, the local stability of the fixed point is not affected by changes in $\sigma^0$ and
there is still a basin of attraction to the fixed point. We thus have coexistence
of attractors.
Figure 2: The value of the steady state price for different values of $f$, the fraction of fundamentalists, in the leverage targeter only model. When $f = 1$ the equilibrium price is equal to the fundamental value of 3.333, and as $f$ decreases the equilibrium price increases.

$\delta$ does not have an effect on the local stability of the steady state, but does affect the cycles. Figure 4 shows that $\delta$ has a large impact on the amplitude of cycles, with cycles roughly having higher amplitude as $\delta$ decreases. This is because a larger $\delta$ means that the volatility estimate puts higher weight on recent observations. Thus a boom will more quickly result in an increased volatility estimate and hence a crash. For a small $\delta$ it takes a long time for the banks to adjust their volatility estimate to reflect the long period of increasing prices and hence the boom continues for a long time, leading to a large amplitude crash. When $\delta$ is very small, however, we see that the local stability of the fixed point turns into global stability. For such a small $\delta$ new information has barely any effect on the volatility estimate, so we can almost consider the volatility estimate to be constant for some initial transient phase. A constant volatility estimate leads to a constant price, and constant prices mean that the observations that are very slowly entering the volatility estimate are always that the volatility was zero. As a result the volatility estimate very slowly decreases to zero, as the price very slowly approaches the equilibrium value.

4.3 All banks leverage optimisers

In this section we focus on the case where all banks act as leverage optimisers. If we fix $n_{2,t} = 1$ we find convergence to a fixed point for all values of $f$ and $\delta$ through simulation. We can prove analytically that any such fixed point has to be equal to $(p^*, p^*, \mu^*, \sigma^{2*}) = (\tilde{p}, \tilde{p}, \tilde{p}, 0)$. The point is globally stable, even
Figure 3: Bifurcation diagram in $\sigma^0$ for the model where all banks are leverage targeters.

Figure 4: Bifurcation diagram in $\delta$ for the model where all banks are leverage targeters.
if convergence may take thousands of iterations. In the fixed point the leverage optimiser will not be buying according to its leverage ceiling, instead it will have the optimal demand of 0, which is equal to the floor of its demand function.

Eigenvalue analysis confirms that all eigenvalues are inside the unit circle. Just as in the targeter case, the eigenvalues are $0, 0, 1 - \delta$ and a fourth eigenvalue which appears to be close to but smaller than 1 for all parameter values.

The intuition behind this stability is as follows. If the leverage optimisers are operating at their maximal leverage, they eventually work themselves into a crisis as we’ve seen in Section 4.2. Volatility then spikes so that the maximal allowed leverage drops. Due to the memory parameter $\delta$ this is somewhat persistent. The demand of the optimiser is thus low for a while, which means the price is near the fundamental value. The forecast $\mathbb{E}[p] = \mu$ will in this time of stable pricing also approach the fundamental value.

The precise reasoning for why the prices are stable at the steady state is as follows. When $\mu = p = \hat{p}$ the leverage optimiser (and the fundamentalist) have zero demand, because they believe the asset won’t change in value in the next time-step ($\mu = p$) and they believe the cost for holding the asset will offset exactly the expected dividends ($p = \hat{p}$). When $\mu = p > \hat{p}$ the optimiser again does not expect to make profit from price shifts, but now the cost for holding the asset is larger than the expected dividends. This is because the cost for holding the asset scales with the (now higher) price, while the dividends do not. We will refer to the resulting losses as ‘fundamental losses’ from here on. Since the optimiser faces a short-selling constraint, they thus have zero demand. The fundamentalist wants to short the asset in this price range but has no one to sell to, so it is not possible to reach this situation. When $\mu = p < \hat{p}$ this story is reversed and the optimiser wants to buy the asset. Since the fundamentalist also wants to buy the asset and we have zero outside supply of assets, it is not possible to reach this situation. In the period of stable prices where $p$ and $\mu$ are very close, we will thus inevitably find the leverage optimiser having near-zero demand, while the fundamentalist is shorting the asset until the price reaches the fundamental value. We have then reached the stable fixed point.

5 Model Analysis

In this section we analyse the full model in Section 3.5.1

5.1 Example of a cycle

Figure 5 shows one cycle with a bubble and a crash, while Figure 6 shows repeated cycles and bubbles and crashes. We will begin by describing one cycle here. The blue line is the price, the red one is the expectation of the leverage optimiser ($\mu$). This means that when the red line is above the blue one, the
Figure 5: Example of a cycle for the default parameters in Table 1. The blue line is the price, the red one is the expectation of the leverage optimiser ($\mu$), and the yellow one is the volatility estimate.
The fundamentalist expectation is always equal to the fundamental value, 3.33 in this case. As the price is above 3.33 for the entirety of the cycle, the fundamentalist always has a short position.

The yellow line is the volatility estimate. When this is high, the permitted leverage will be low, and vice versa. For the leverage targeter this directly corresponds to their demand, since they will always buy as much as they are allowed to given the leverage constraint. For the leverage optimiser, it will affect both their optimal demand and the limit on this demand.

The cycle begins at the end stages of a period of moderation. The memory of the last big crisis is fading, and the banks estimate volatility to be low. As a result, the leverage targeter sees that its target is going up, and it starts to increase its demand. Meanwhile, the leverage optimiser, which before believed the asset to be underpriced, sees the price pick up quickly. Their expectation does not increase as quickly so that they now believe the asset to be overpriced. Their demand is thus 0, while the leverage targeter demand is large and growing. The price is also above the fundamental, so that the fundamentalist has a negative demand. This demand however is a whole lot smaller than that of the targeting banks (e.g. because they cannot leverage their investments). As a result, the price of the risky asset is blowing up and the leverage targeter with the large demand is making big profits. Since the more hesitant leverage optimiser strategy is not making any profits, more and more agents change from optimisers to targeters, which further fuels the boom.

Eventually, however, the banks will see that the price has been increasing very rapidly for a while, which will mean that their estimate of volatility will start increasing. When the volatility increases, the banks will have a decreasing leverage maximum, meaning the leverage targeters have to decrease their risky asset holdings a bit. This results in a decreased price. Since, however, the leverage targeters have massive leverage at this stage, this decrease in price results in very large losses. When the leverage targeters make losses, the banks start to shift strategies to the leverage optimising strategy, which has a much lower (initially zero) demand. This shift thus leads to further price drops. At the same time, the dropping price increases the volatility estimate further, decreasing the leverage target further. We thus have two sources of positive feedback leading to a rapid crash.

\[4\text{Recall from Section 4.3 that this does not directly translate into a positive or zero (due to short-selling constraint) demand. For example, when the red line is only a bit above the blue line and both are far above the fundamental value, demand may still be zero. This is because the (expected) losses due to the difference in cost and dividends may outweigh the profits due to increased price. When the lines are a bit further apart however, as in this example, we do see their relative position translate directly into positive or zero demand.}\]
When these losses are sufficiently large almost every bank switches strategies to the leverage optimiser. The price drops so quickly that the leverage optimiser actually sees the asset as underpriced ($\mu > p$), and wants to increase its demand. Thus their demand comes close to that of the leverage targeter, which leads to the rapid price drop suddenly halting. Since the two demands are now close together they make similar losses, and the banks split across the two strategies nearly equally.

We now enter a period of relative stability, which we will refer to as the ‘great moderation’ from here on out. The leverage targeter has an increasing demand due to a monotonic decreasing of the volatility estimate, but the leverage optimiser has a demand that’s decreasing at roughly the same rate. This demand decreases because the price and their expectation of the price come closer together, meaning that they see less and less opportunity for profits. This outweighs the reduction in volatility.\(^5\) The net demand by the banks (who are about equally split over the two strategies) stays roughly constant close to zero and with the fundamentalist shorting the asset this results in a slowly decreasing price. This continues until the leverage optimiser hits the lower bound of its demand, the short selling constraint. Their demand can then no longer decrease to compensate for the increasing demand of the leverage targeter, and the price shoots up.

The cycle thus repeats, although the different booms and crashes are of different sizes and lengths, and after a few smaller ones a bigger one is likely to occur because the volatility estimate will be smaller at the start of the cycle. When there is a particularly rapid crash and the leverage optimiser is faced with their demand ceiling, it aggravates the crisis even more. Normally, the dropping demand of the leverage targeter is somewhat compensated for by an increased demand for the leverage optimiser, who see the price fall below their expectation for the price (red line) and thus want to invest. When their demand, however, cannot grow further due to the volatility estimate being so high that they hit their ceiling, the price can drop much further. An extra large crash occurs, which compensates for the slight upward trend that can be seen in the cycle in this example. There is thus, over time, no upward or downward trend in the price. Its distribution appears to be stationary. Figure 6 shows these bigger crashes and the stationarity overtime.

\(^5\)This is very hard to see in the picture since volatility is at a different scale from the blue and orange line.
Figure 6: Simulation with deterministic dividends. The blue line is the price, the red one is the expectation of the leverage optimiser ($\mu$), and the yellow one is the volatility estimate.

### 5.2 Fixed point

We will proceed to analyse a potential fixed point of the model, as well as its uniqueness and stability. As we can see from the following calculation, a fixed point exists and it is unique:

$$p'^* = p^*$$

$$(1 - \delta)\sigma^{2,*} + \delta \log \left( \frac{p^*}{\hat{p}^*} \right) = \sigma^{2,*}$$

$$(1 - \delta)\sigma^{2,*} + \delta 0 = \sigma^{2,*}$$

$$(1 - \delta)\sigma^{2,*} = \sigma^{2,*}$$

So that either $\delta = 0$ or $\sigma^{2,*} = 0$.

$$(1 - \delta)\mu^* + \delta p^* = \mu^*$$

$$\delta p^* = \delta \mu^*$$

So that either $\delta = 0$ or $p^* = \mu^*$.

Now to find $p^*$, we begin by analysing in which of the three segments of leverage optimiser demand the fixed price would be. The leverage optimiser, in a
fixed point, has zero demand if $p^* \geq \tilde{p}$ and a positive demand if $p^* < \tilde{p}$. The leverage targeter has a positive demand for all $p^*$. The fundamentalist has a negative demand if $p^* > \tilde{p}$ and a positive demand if $p^* < \tilde{p}$ (and zero for $p^* = \tilde{p}$).

Therefore, if $p^* < \tilde{p}$, the optimiser would have positive demand, the targeter would have positive demand and the fundamentalist would have positive demand. Thus everyone would be buying, while the supply is zero, and there is thus no equilibrium price. We therefore reach a contradiction, and conclude that if an equilibrium price $p^*$ exists, it must hold that $p^* \geq \tilde{p}$. This means that leverage optimiser demand in equilibrium would be zero.

To actually determine the equilibrium price, we need to simultaneously solve for the equilibrium fractions of leverage targeters and optimisers. Since the optimiser demand in equilibrium is zero, they will make zero profit. The targeter however would be making a loss in equilibrium if $p^* > \tilde{p}$ (the aforementioned 'fundamental loss').

We can write the equilibrium fraction of leverage targeters as:

$$n^*_{1} = \frac{e^{\beta \frac{\alpha \tilde{p}}{\sqrt{\sigma^2 p^*}} (p^* + \tilde{y} - R \tilde{p})}}{e^{\beta \frac{\alpha \tilde{p}}{\sqrt{\sigma^2 p^*}} (p^* + \tilde{y} - R \tilde{p})} + 1}$$  \hspace{1cm} (12)

And the equilibrium price is thus the solution to:

$$p^* = \frac{\tilde{p}}{2} + \sqrt{\frac{\tilde{p}^2}{4} + \frac{a \alpha E (1 - f)}{f R \sqrt{\sigma^2}} \frac{e^{\beta \frac{\alpha \tilde{p}}{\sqrt{\sigma^2 p^*}} (p^* + \tilde{y} - R \tilde{p})}}{e^{\beta \frac{\alpha \tilde{p}}{\sqrt{\sigma^2 p^*}} (p^* + \tilde{y} - R \tilde{p})} + 1}}$$  \hspace{1cm} (13)

Numerical plots of the implicit function for the benchmark parameters show that the fixed point is unique.

The price in equilibrium is essentially always larger than $\tilde{p}$, again with the exception of $f$ tending to 1, $E$ tending to 0, etc (as in Section 4.2). Since for $p^* > \tilde{p}$ the targeter is making a loss in equilibrium while the optimiser is making zero profit, $n^* < 0.5$ for most parameter values.

When $\delta = 0$ we have a continuum of fixed points since $\sigma$ and $\mu$ will always be equal to their initial value.

### 5.3 Stability analysis

Since we cannot write down the fixed point analytically, we cannot perform a stability analysis in full generality. We can however calculate the Jacobian and find its eigenvalues for specific parameter values. We can also use bifurcation diagrams to see how the dynamics depend on parameters more broadly.
We can say in generality that the eigenvalues of the Jacobian are: 0, one repeated eigenvalue of $1 - \delta$ and two other eigenvalues, which are sometimes a complex pair and sometimes two positives real eigenvalues. Since $0 \leq \delta \leq 1$ it is only in these last two eigenvalues that we see bifurcations. We will consider these for some of our parameters now.

### 5.3.1 Intensity of choice

For our default parameters (see Table 1) we find that there are no values of $\beta$ for which we have global stability. While it is not entirely monotonic, we find that increasing $\beta$ reduces the equilibrium price and increases the amplitude of cycles (see Figure 7).

The reduced equilibrium price is because $\beta$ represents the intensity of choice when banks pick their strategy. The targeting strategy has positive demand in equilibrium and makes fundamental losses, while the optimising strategy has zero demand and makes no losses. When the intensity of choice is higher, more banks will choose the optimising strategy in equilibrium, and thus more banks will have zero demand. This leads to a lower price.

The increased amplitude of cycles is due to a more pronounced herding effect at the top and bottom of a cycle. For example, when all banks suddenly switch to the leverage optimising strategy during a crash, while this strategy still has zero demand, the price can fall a lot further than it would if a significant number of banks were still using the targeting strategy.

We also find that for larger $\beta$ we start seeing extra large period of moderation (that we could call ‘depressions’) occur. Any period of moderation is longer with a larger $\beta$ because the fraction of leverage optimisers is larger during the moderation for a higher $\beta$, meaning that the leverage targeter has less influence on the price and the optimiser’s demand drops slower and reaches the short selling limit later. Occasionally this can take so long that the volatility reaches a particularly low value. Then $\frac{\mu}{\alpha(\sigma^2 + \sigma^0)}$ can start outweighing $\frac{\mu + Rp}{\alpha(\sigma^2 + \sigma^0)}$ which leads to a bump in the optimiser’s demand. This bump ends when $\sigma^2$ becomes lower than $\sigma^0$ so that the effect of volatility stabilises. Then the optimiser demand starts decreasing again since $\mu$ continues to approach $p$. Meanwhile the fraction of leverage targeters has dropped so low that they have almost no effect on the price anymore, so that it takes an extremely long time for the leverage optimiser to drop its demand back down to the short selling limit. As a result we go through a phase where $n_1$ approaches zero and the price is decreasing slowly for a long time. Eventually the limit is reached and we go back into cyclical behaviour until another depression occurs. Based on eigenvalue analysis, the fixed point is locally stable only for $\beta < 0.00245$. This is not visible in the bifurcation diagram due to the value being so small. For this range of values we have coexistence of attractors: we can find chaotic cycles or convergence to the fixed point depending on the initial conditions. At about 0.00245, the two
leading eigenvalues which are complex cross the unit circle: a Hopf bifurcation. We find a positive Lyapunov exponent for all $\beta$.

5.3.2 Memory

For our default parameters there are no values of $\delta > 0$ for which the fixed point is locally stable. We find a positive Lyapunov exponent for all $0 \leq \delta \leq 1$. As can be seen from Figure 8, the amplitude of cycles is now increasing in $\delta$. This is notable since the amplitude was decreasing in $\delta$ when we only considered the leverage targeter.

Recall from Section 4.2 that smaller $\delta$ led to larger booms since it would take longer for the banks to realise they were in a boom. In our full model, we do not see this effect because the boom happens much quicker. This acceleration happens for two reasons: first of all because the boom is preceded by a period of ‘great moderation’ so that the previous boom/bust has already nearly been forgotten by the time the next boom starts. And secondly because the sudden mass switching of strategies from optimiser to targeter during the boom adds a positive feedback loop, thus fuelling the boom.

For large $\delta$, we see slightly larger amplitude cycles for the full model than for the leverage targeter only model based on the bifurcation diagram. This is actually only due to an occasional large cycle. In between these large cycles, the full model cycles have a reduced amplitude due to the existence of the leverage optimiser. As we saw in Section 5.1, the leverage optimiser can reduce the size of the crash because it starts believing the asset is undervalued somewhere midway through the “leverage targeter’s crash”. But, after a number of reduced amplitude cycles, the volatility estimate can reach exceptionally low levels. This leads to exceptionally high leverage during the boom. We then reach a ‘perfect storm’ when the leverage optimisers hit their demand ceiling during a crash (meaning all banks are effectively leverage targeters). We thus see most banks acting like leverage targeters during the entirety of the cycle, but the leverage target being chased is larger than it ever is in the leverage targeter only model due to the initial low volatility estimate.

5.3.3 Fraction of fundamentalists

For $f = 1$ we have a globally stable fixed point at the fundamental price, since this is the case from section 4.1. For values of $f$ below 0.99998 (and our default parameters) we have a locally unstable fixed point, with the price above the fundamental, increasing as $f$ decreases. This is because the higher the number of banks is, the higher the positive demand in equilibrium. The amplitude of the cycles increases as $f$ decreases, since prices change more based on changes in the bank's demand if the fraction of fundamentalists is smaller. The Lyapunov exponent is positive for all values of $f$ except 1.
5.3.4 Floor volatility estimate

Figure 9 shows a bifurcation diagram in $\sigma^0$. For $\sigma^0 \geq 0.029$ we find that the fixed point is globally stable. The leverage targeter demand is capped at such a low demand that their destabilising effect is curbed completely. For $\sigma^0 < 0.029$ the fixed point is locally unstable, and the Lyapunov exponent is positive. The bifurcation occurring here is Hopf, with a pair of complex eigenvalues crossing the unit circle.

The equilibrium price increase as $\sigma^0$ decreases. This is because the leverage targeter operates at its demand ceiling (as determined by the inverse of $\sigma^0$) in equilibrium, so when it has a higher ceiling their demand is higher and so is the price.

The amplitude of cycles increases as $\sigma^0$ decreases as well. When $\sigma^0$ is smaller, the leverage targeter can buy more during the boom, which results in larger price spikes.

![Bifurcation diagram](image)

Figure 7: Bifurcation diagram for $\beta$. Hopf bifurcation occurs at $\beta = 0.00245$. Other parameters as in Table 1.
Figure 8: Bifurcation diagram for $\delta$. Other parameters as in Table 1.

Figure 9: Bifurcation diagram for $\sigma^0$, showing a Hopf bifurcation at 0.029. Other parameters as in Table 1.
5.4 Leverage

Figure 10 shows a plot of security broker/dealer leverage and the VIX volatility index for the period from 1980-2019. Figure 11 shows leverage and estimated volatility as produced by our model. We note that the data shows two booms and crashes in a 30 year time period. The measure of leverage peaks right before the VIX does. The data generated by our model also shows two cycles in a 30 year period. Secondly, we find that again the leverage peaks right before the measure of estimated volatility does. Another feature of the real data that we manage to replicate is the relationship between leverage and estimated volatility. We find that the correlation between $\log(\text{leverage})$ and $\log(\text{estimatedvolatility})$ in the long run (1000 iterations) is $-0.5805$, thus very close to the empirical value found in Adrian and Shin (2013), which is between $-0.3$ and $-0.5$ depending on the specification.

Figure 10: Plot of Security broker/dealer leverage and the VIX volatility index as reported by the St. Louis Federal Reserve. Quarterly data.
6 Choice of strategy

In this section we look at the fraction of banks that chooses the leverage targeting strategy ($n_1$). We consider whether this fraction can converge to an extreme value, and we discuss to what extent the choice of strategy by the banks is justified ex post, when we see how well the strategies actually performed.

6.1 Both strategies persist

While the discrete choice probability function that determines $n_1$ can never take on the values 0 or 1, it could be possible for one strategy to be pushed out of the market almost entirely over time, so that $n_1$ would tend to either 0 or 1 over time. Here we discuss why this is not actually possible for the two strategies in our market.

Suppose the leverage targeters were pushed out of the market (almost) entirely, so that $n_1$ would be tending to 0. From 4.3 we know that in this case, the price in the market would be tending to the fundamental value, and the volatility would be dropping to 0. As this happens, however, the leverage targeter strategy demand would start exploding. Even if only a very small fraction of leverage targeters were present when this happened, this would have an increasing effect on the price. This small increase is magnified by the huge position of the leverage targeter strategy, meaning that this strategy will make a big profit. As a result, some banks that were using leverage optimising would switch to leverage targeting, and $n_1$ would increase. This contradicts the assumption that $n_1$ tends to 0.
Suppose the leverage optimisers were pushed out of the market (almost) entirely, so that $n_1$ would be tending to 1. From 4.2 we know that in this case, either we reach a fixed point (far above the fundamental value) or we enter leverage cycles. Assume first that we reach a fixed point. When the price stabilises, the higher this price is, the lower the resulting $n_1$. This is because for an (almost) stable price above the fundamental value, the leverage optimiser has zero demand while the leverage targeter is making fundamental losses. These losses are bigger the higher the stable price is. Thus, a higher price results in a lower $n_1$, and for a price as high as the fixed point for leverage targeter only, this $n_1$ would be well below 0.6. Thus $n_1$ no longer tends to 1. If we enter instead leverage cycles, then inevitably during the crashes in these cycles, the leverage targeters would make big losses. We know that during these crashes, the losses by the leverage optimiser are not as big, thus resulting in them performing better and banks switching to their strategy. Again, $n_1$ no longer tends to 1.

6.2 Realised profits

We can consider now to what extent these results on the use of the strategies make sense ‘ex post’. By this we mean, whether when profits have been realised, it appears that banks chose the ‘right’ strategy beforehand.

The leverage targeter makes positive profits over time for our standard parameters. This is because they have a bigger position during the boom than during the bust, so that they benefit more from the rising prices than they lose from the dropping prices. This difference is so big that it outweighs the losses the targeter makes during the great moderation, when they take an increasingly big position while the price is slowly dropping.

The leverage optimiser on the other hand makes losses over time. They don’t benefit from the boom since they have zero demand during it, but they do lose in the bust because they start taking a position when the price drops. When a particularly large crisis has happened and the price makes a recovery the optimiser makes a small profit. During the moderation they make small, decreasing, losses since the price is dropping and their positive demand is dropping. Overall the optimiser thus performs worse than the leverage targeter.

6.2.1 Why the average fraction stays at roughly 0.5

The difference in profits is not reflected in the choice of strategy. The average fraction of leverage targeters, $n_1$, is 0.513 for our standard parameters. Figure 12 shows a time series of the fraction of leverage targeters and the price. During the boom the fraction of targeters jumps to 1, since the targeter is making huge profits and the optimiser is making nothing. During the bust the fraction of targeters drops to close to 0 since the targeter losses are much bigger than the optimiser losses due to their large leveraged position. During the great
moderation it drops slowly down from about 0.5, compensating for the fraction not dropping all the way to 0.  

6.2.2 How our key parameters influence this

$\delta$ leads first to increasing fraction of leverage targeters and then later, non-monotonically, to a decreasing fraction. This is because there are two opposing forces. One is that as $\delta$ increases, booms and crashes occur more regularly so that the leverage targeter has more regular opportunities to make profits. The other is that when $\delta$ increases, the booms (and crashes) become shorter in duration and hence have lower amplitude, so that the leverage targeter makes smaller profits from them.

$\beta$ initially has very little effect on the fraction of leverage targeters. This is because both the move to leverage targeting and the move to leverage optimising happen more quickly, which balances each other out. When the periods of great depression start to occur, however, the fraction of leverage targeters decreases dramatically. This is because the leverage targeting strategy is barely represented during the long periods of depression.

The average fraction of targeters is increasing in $f$. This is because the fundamental losses that are decreasing in $f$ are more significant for the leverage targeter than for the leverage optimiser because the targeter takes bigger positions.

$\sigma^0$ increasing leads first to a drop in the fraction of leverage targeters and then to an increase. This is because at first, the increase in $\sigma^0$ means that

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6When profit memory is included in the choice of strategy, this has very similar results to increasing intensity of choice parameter $\beta$. 

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the leverage targeter is leveraging less, thus making less profits. After a while however another effect takes over. The higher $\sigma^0$ leads to lower average prices, thus meaning the leverage targeter makes less fundamental losses. The leverage optimiser meanwhile inititally benefits from less pronounced booms and crashes. Later however $\sigma^0$ becomes so big that the optimiser’s optimal demand drops lower and lower, thus allowing for less opportunity to make any profits.

7 Shortselling allowed

7.1 Model set-up

In this section we investigate how the results would change if we allowed the banks to short sell. It is not clear how this should be implemented for the leverage targeting strategy, since they do not form an expectation on the future price of the asset, but only on the size of the exposure they would like to have to it. While it would be possible to have them form an opinion simply on whether the asset will go up or down in price, they would then be able to make a very sudden swap from a huge positive to a huge negative exposure. This would create a large discontinuity in the demand function, resulting in very erratic behaviour that does not seem realistic. We therefore do not allow the leverage targeter to short-sell even in this case, and only allow the optimiser to short-sell.

To implement the short selling, we do need to take into account that a short position is still a position and is thus still constrained by the leverage control policy. We thus replace the demand function of the leverage optimiser by the following:

$$z_{2,t} = \min \left( \max \left( -\frac{\alpha E}{\sqrt{\sigma_t^2 + \sigma^0 p_t}}, \frac{\mu_t + \bar{y} - Rp_t}{a \sigma_t^2}, \frac{\alpha E}{\sqrt{\sigma_t^2 + \sigma^0 p_t}} \right), \frac{\alpha E}{\sqrt{\sigma_t^2 + \sigma^0 p_t}} \right)$$

This demand function is no longer monotonic in $p_t$. Unfortunately, this monotonicity was what allowed us to write down the equilibrium price analytically. Now we can not do this, but we can solve for the price numerically, allowing for simulation results.

7.2 Model dynamics

Introducing short-selling for banks using the leverage optimising strategy stabilises the dynamics. In the very initial phase it is possible for exceptionally large fluctuations to occur, but as soon as the ‘great moderation’ begins, no more cycles occur. This is because in the original model, the end of the great moderation occurs when the leverage optimiser reaches their short-selling limit. When the optimiser can short-sell, we see their decreasing demand compensate for the increasing demand of the leverage targeter. Eventually the leverage targeter reaches the ceiling of its demand as dictated by $\sigma^0$, because the volatility
estimate has effectively reached 0. They now have a constant very positive demand. The leverage optimiser has a very negative demand, but it is a bit smaller than that of the leverage targeter. This is because their optimal demand includes the inverse of $\sigma^0$ rather than $\sqrt{\sigma^0}$, leading to a smaller absolute value demand. The targeter and optimiser together thus have a bit of excess demand, which means the fundamentalist is supplying by shorting the asset. The equilibrium price is therefore still above the fundamental value (see Section 5.2), although it is much less elevated than we saw in our full model.

8 Conclusion

We developed a model where banks can choose between leverage targeting and leverage optimising every time step. We find that both strategies persist and leverage cycles occur. While the leverage targeting strategy makes positive and the leverage optimising strategy makes negative profits over time, both strategies are about equally prevalent. This is due to the switching mechanism. During the boom the targeter makes huge profits and the optimiser makes no profits, so almost all banks switch to targeting. During the bust the targeter makes losses, but smaller losses than they made profits during the boom because they have a smaller position. Meanwhile the optimiser makes small losses. The leverage optimiser strategy is thus more tempting, and almost all banks switch to leverage optimising.

$\delta$ is the main driver of the chaotic cycles. It is the parameter that ensures that the leverage targeter does not realise it is in a boom when the boom starts, allowing it to continue. This booming behaviour by the leverage targeter is the main reason cycles occur. While setting $\delta$ equal to 0 still allows for some cyclical behaviour due to $\beta$, these are perfectly predictable 4-cycles. $\beta$ is hence also a contender for most important driver, but again, $\beta$ being equal to 0 does not remove the cycles because $\delta$ still makes the cycles occur. The dynamics are very robust under changes to $\delta$, on the standard range of $0 \leq \delta \leq 1$ no bifurcations occur. $\delta$ is also the main parameter controlling the frequency of crises.

We calibrated our parameters to replicate the range of leverage observed in the data as well as the frequency of crises occurring. Our model reproduces the correlation between the logarithms of leverage and estimated volatility as found in Adrian and Shin (2013). Our model also reproduces the spike in volatility following a spike in leverage.

Allowing short selling for the leverage optimiser removes the leverage cycles all together. This is because the leverage optimiser can reduce its demand unlimitedly to compensate for the increasing demand by the leverage targeter. It leads to a stable price slightly above the fundamental value.
References


Appendix

A.1 Details Model Derivation

A.1.1 The fundamentalist trader

The demand of the fundamentalist trader closely follows Brock and Hommes (1998). Our fundamentalist does not borrow cash, but they may borrow stocks to engage in short selling. Assume that their wealth at time $t$ is equal to $W_t$. First, we look at the case where they go long rather than short on the risky asset. In that case they will not borrow anything. If they invest $C_t$ of their wealth into the risk free asset and purchase $z_t$ units of the risky asset for a price of $p_t$, their wealth at time $t + 1$ will be equal to:

$$ W_{t+1} = W_t + rC_t + \bar{y}z_t + (p_{t+1} - p_t)z_t $$  \hfill (15)

If, instead, the trader would short the risky asset, so that their holdings $z_t$ are negative, their wealth at time $t + 1$ will be:

$$ W_{t+1} = W_t + rC_t + \bar{y}z_t - rp_tz_t - (p_t - p_{t+1})z_t + rp_tz_t $$  \hfill (16)  

$$ = W_t + rC_t + \bar{y}z_t + (p_{t+1} - p_t)z_t $$  \hfill (17)

where the first term represents the initial wealth and the second the income from shorting the risky asset. The other terms: in short selling you borrow and immediately sell on a number of assets, meaning that they appear on the liabilities side rather than asset side of your balance sheet. You then hold cash from the sale on your balance sheet, which can earn you interest. Then, at the next time step, you buy back the assets you sold in order to pay back your lender. You also owe this lender any dividends that this asset would have earned in the time step, as well as some interest. The third term represents these dividends you owe, the fourth the interest you earned on the cash you were holding, the fifth the earnings or losses on the sale and subsequent purchase of the assets, and the final term the interest you pay on the stock loan.

Now we see that these two expressions are in fact equal, and since $C_t = W_t - z_t p_t$ in both cases we have:

$$ W_{t+1} = RW_t + (p_{t+1} + \bar{y} - Rp_t)z_t $$  \hfill (18)
If we now assume that our fundamentalist trader is a myopic mean variance wealth maximiser with risk aversion parameter $\bar{a} > 0$ their demand for shares $z_t \in \mathbb{R}$ solves:

$$\max_{z_t} \left( E_t W_{t+1} - \frac{\bar{a}}{2} \text{Var}_t W_{t+1} \right) = \max_{z_t} \left( E_t p_{t+1} + \bar{y} - R p_t - \frac{\bar{a}}{2} \text{Var}_t p_{t+1} \right)$$

Now we assume that $E_t p_{t+1}$ and $\text{Var}_t p_{t+1}$ are both constant in time for our trader, which is the feature that earns them their fundamentalist title. Their expectation is always that the price of the risky asset will return to its fundamental value $\bar{p} = \frac{\bar{y}}{r}$, and they always think the variance of the risky asset is $\sigma$. By letting $a = \bar{a} \sigma$ we then find the demand of the fundamentalist trader to be:

$$z_t = \frac{\bar{p} + \bar{y} - R p_t}{a}$$

A.1.2 The bank

Our bank is an investor who is allowed to borrow cash to leverage their investments, but is not allowed to short sell. Let us first consider how borrowing cash to leverage investments would affect the development of their wealth. When the bank borrows cash to purchase assets, we have $C_t < 0$ and $z_t > 0$.

$$W_{t+1} = W_t + r C_t + \bar{y} z_t + (p_{t+1} - p_t) z_t$$

where $C_t$ is negative so that the second term represents the interest paid on the cash loan received. We see that this is equal to the wealth in (15). If the bank would thus not face a leverage constraint, and we assume they are also myopic mean variance maximisers with risk aversion parameter $a$, their demand would be equal to

$$\max_{z_t \geq 0} \left( E_t p_{t+1} + \bar{y} - R p_t - \frac{\bar{a}}{2} \text{Var}_t p_{t+1} \right)$$

The solution to which is:

$$z_t = \max \left( \frac{E_t p_{t+1} + \bar{y} - R p_t}{a \text{Var}_t p_{t+1}}, 0 \right)$$

However, the bank does in fact face a leverage constraint.

A Value at Risk (VaR) constraint is one so that portfolio losses have at most probability $c$ to exceed some dollar value. To calculate the VaR, our banks need to have some belief on the distribution of their risky assets. We assume that banks believe that stock returns are normally distributed with zero mean (Aymanns and Farmer (2015)). The variance of the stock returns is estimated through an exponential moving average. Given the volatility estimate of returns on one dollar of the risky asset is $\sigma_t$, we can derive the value at risk per dollar
worth of asset. We want that the probability that you lose more than VaR dollars is no bigger than 1-c. Assuming you have one dollar of assets at time \( t \), so that \( z_t p_t = 1 \) and \( z_t = \frac{1}{p_t} \), we have that:

\[
\begin{align*}
P(p_{t+1} z_t \leq p_t z_t - VaR) &\leq 1 - c \\
P\left(\frac{p_{t+1}}{p_t} \leq 1 - VaR\right) &\leq 1 - c \\
P\left(\frac{p_{t+1} - p_t}{p_t} \leq -VaR\right) &\leq 1 - c
\end{align*}
\] (21)

Now we have assumed that stock returns, and thus \( \frac{p_{t+1} - p_t}{p_t} \), are normally distributed with mean 0 and variance \( \sigma_t \). The left hand side of (21) is thus equal to:

\[
\frac{1}{2} + \frac{1}{2} \text{erf}\left(\frac{-VaR}{\sqrt{2}\sigma_t}\right)
\]

Which means that:

\[
VaR = \sqrt{2}\sigma_t \text{erf}^{-1}(2c - 1)
\]

Now this is the Value at Risk for one dollar of the risky asset. To find out the total Value at Risk, we need to multiply by the dollar amount of risky assets the bank holds. The total value at risk then cannot exceed the equity of the bank (\( E \)). The value at risk constraint thus looks like:

\[
VaR z_t p_t \leq E
\]

So that the leverage, \( \frac{\text{Assets}}{\text{Equity}} \) is bound by:

\[
\frac{\text{Assets}}{\text{Equity}} = \frac{z_t p_t}{E} \leq VaR^{-1} = \frac{\alpha}{\sigma_t}
\]

Where \( \alpha = \left(\sqrt{2}\text{erf}^{-1}(2c - 1)\right)^{-1} \)

Aymanns and Farmer (2015) have two versions of their model: one with constant and one with variable equity. In the case with constant equity any profits just ’disappear’ (for example, they may be directly distributed to equity holders), this is motivated by the empirical findings in Adrian and Shin (2010). In the other case, where equity is variable, there is a mechanism to distribute equity between the traders to prevent any agent blowing up to take over the market completely. Since this makes the model more complicated, and we are already significantly complicating the model, we choose here to stick with a constant equity \( E \).

The estimation of the variance of returns on the risky asset is as in Aymanns et al. (2016). The estimate depends on the estimate of variance in the previous timestep, the most recently experienced returns and memory parameter \( 0 < \delta < 1 \). It is an exponential moving average:
\[ \sigma_t^2 = (1 - \delta)\sigma_{t-1}^2 + \delta \log \left( \frac{p_t}{p_{t-1}} \right)^2 \]

So with the volatility estimate, \( \alpha \) and the equity a given, we have the maximal demand of the bank as a function of price, namely:

\[ z_t \leq \frac{\alpha E}{\sigma_t p_t} \tag{22} \]

Now in Aymanns and Farmer (2015), the bank simply buys as much of the risky asset as it is allowed to given this restriction. They will thus always have demand:

\[ z_t = \frac{\alpha E}{\sigma_t p_t} \]

In this paper, we let the bank choose between two different strategies. It either simply has this maximal demand, or it chooses a more sophisticated strategy. This second option entails forming an expectation of the price of the risky asset in the next timestep, \( E_t p_{t+1} \), and actually considering what the optimal demand would be given this expectation of the price. If this demand is lower than the maximal demand allowed given the leverage constraint, the bank will purchase the lower amount. The bank can thus choose between ‘blindly’ taking as much risk as is allowed, or considering whether they actually expect to profit from taking this amount of risk. We will investigate which of these two strategies wins in the long run under which circumstances. The banks can either choose the leverage targeting strategy, which gives them demand \( z_{1,t} \):

\[ z_{1,t} = \frac{\alpha E}{\sigma_t p_t} \]

or the leverage optimising strategy with demand \( z_{2,t} \), which is a combination of (21) and (22):

\[ z_{2,t} = \min \left( \max \left( 0, \frac{E_t p_{t+1} + \bar{y} - Rp_t}{a\sigma_t^2} \right), \frac{\alpha E}{\sigma_t p_t} \right) \tag{23} \]

Now we still need to establish how banks using the leverage optimising strategy form their expectation of the price of the risky asset, \( E_t p_{t+1} \). This can remain a flexible part of the model, and we can even examine how different methods of expectation formation affect the results. As a base model, however, we use a method similar to that used for estimating the variance, and similar to that in Danielsson et al. (2004).

\[ E_t p_{t+1} = \mu_t = (1 - \delta)\mu_{t-1} + \delta p_{t-1} \]

This is in essence a moving average, and a proxy for the fundamental value of the asset with consideration of animal spirits. The difference between this and Danielsson et al. (2004) is that they replace the \( p_{t-1} \) in here by returns on the risky asset, meaning they are also predicting returns.
The way in which the bank chooses between the two strategies is based on
the heuristic switching model (Brock and Hommes (1997)). First, the profits
realised by each strategy in the last time step are calculated.

For \( h \in \{1, 2\} \), the profits realised are equal to:

\[
\pi_{h,t} = z_{h,t-1}(p_{t+1} + \bar{\gamma} - Rp_t)
\]  

(24)

Note that these profits are not added to the equity of the bank. They are,
for example, immediately handed out to equity holders.

Finally the fraction of banks which use the leverage maximising strategy is:

\[
n_{1,t} = \frac{e^{\beta \pi_{1,t-1}} - 1}{e^{\beta \pi_{1,t-1}} + e^{\beta \pi_{2,t-1}}}
\]

And \( n_{2,t} = 1 - n_{1,t} \)

Now we also have a fixed fraction of fundamentalist traders, \( f \), so the actual
fraction of traders which are both bank and use leverage maximisation is \((1 - f) n_{1,t}\) and similarly for leverage optimisation.

The price of the risky asset is then determined by equating the total demand
to the total supply, which we set equal to 0. There is a slight complication here,
since in reality when determining how much to actually go purchase on the
market now, it should be taken into account that there are currently already
holdings. We justify this as follows:

\[
\sum_i z_{i,t}(p_t) - z_{i,t-1}(p_t) = 0
\]  

(25)

\[
\sum_i z_{i,t}(p_t) = \sum_i z_{i,t-1}(p_t)
\]  

(26)

and \( \pi_i \) are those profits that we make disappear out of the system, so in effect
set equal to zero. \( \sum_i z_{i,t-1}(p_{t-1}) = 0 \) and thus we can consider \( \sum_t z_{i,t}(p_t) = 0 \).

We thus have the following supply-demand equation:

\[
f z_{f,t} + (1 - f) n_{1,t-1} z_{1,t} + (1 - f) n_{2,t-1} z_{2,t} = 0
\]

Which is solved for \( p(t) \). To do this we separate (6) to find out the three
possible prices the demand-supply equation may produce. These are:

\[
p_{0,t}^* = \frac{\bar{\gamma}}{2} + \sqrt{\frac{\bar{\gamma}^2}{4} + 4acn_{1,t-1} \frac{E}{\sqrt{\sigma_t^2 + \sigma^2}} (1 - f) Rf}
\]  

(28)

\footnote{\textsuperscript{7}}

Now to assess the performance of each strategy, the banks can include some memory of
previous profits. The performance measures \( U_h \) then are as follows:

\[
U_{h,t} = \pi_{h,t} + \eta U_{h,t-1}
\]

However, for the vast majority of analysis we set \( \eta = 0 \). This allows for a fixed point and
reduces the number of dimensions in our model.
\[
P^*_t = \frac{R\hat{p} + (1-f)n_{2,t-1}^{\left(\mu_t + \tilde{y}\right)}}{\sigma_t^2 + \sigma^2} + \frac{\left(R\hat{p} + (1-f)n_{2,t-1}^{\left(\mu_t + \tilde{y}\right)}\right)^2 + 4\left(R + \frac{R(1-f)n_{2,t-1}^{\left(\mu_t + \tilde{y}\right)}}{\sigma_t^2 + \sigma^2} + \frac{\alpha E}{\sigma_0} - 1\right)}{2\left(R + \frac{(1-f)n_{2,t-1}^{\left(\mu_t + \tilde{y}\right)}}{\sigma_t^2 + \sigma^2} + \frac{\alpha E}{\sigma_0} - 1\right)}
\]

\[
p^*_C,t = \frac{\hat{p}}{2} + \sqrt{\frac{\hat{p}^2}{4} + \frac{\alpha E \left(1-f\right)}{\sqrt{\sigma_t^2 + \sigma^2} R}}
\]

(29)

Now since \(n_1(t) \leq 1\) we have that \(\tilde{p}_0(t) \leq p^*_C(t)\). This allows us to establish that the price, which is always the ‘middle’ bit of the three sections of (6), is equal to:

\[
p_t = \min(\max(p^*_0, p^*_\mu), p^*_C)
\]

(31)

### A.2 Price fixed point

\[
z^*_f = \frac{\hat{p} + \tilde{y} - Rp^*_0}{a}
\]

\[
z^*_1 = \frac{\alpha E}{\sigma_0^2 p_t}
\]

\[
z^*_2 = \min\left(\max\left(0, \frac{\mu^*_t + \tilde{y} - Rp^*_0}{a\sigma_0^2}, \frac{\alpha E}{\sigma^* p^*}\right), \frac{\alpha E}{\sigma_0^2 p^*_0}\right)
\]

Now since \(p^*\) is not necessarily \(\frac{\tilde{y}}{R}\) it is not immediately clear in what section the demand of \(z^*_2\) will be. We can go through the same steps as we did before, finding the three different points of crossing for the three sections of the demand function and write the equilibrium price as a nested min/max function. This equilibrium price is only a function of the parameters and the fractions.

As an example, this is how we find \(p^*_0\) from equating supply and demand:

\[
f\frac{\hat{p} + \tilde{y} - Rp^*_0}{a} + (1-f)0.5\frac{\alpha E}{\sigma_0 p^*_0} + 0.5(1-f)0 = 0
\]

\[
f(\hat{p} + \tilde{y})p^*_0 - f R(p^*_0)^2 + \frac{a(1-f)0.5\alpha E}{\sigma_0} = 0
\]

we then apply the quadratic formula to find \(p^*_0\). We use the same method to find the other two points of crossing.
A.3 Stability Analysis

A.3.1 Leverage targeter only

\[
J = \frac{\partial (g_{p'}, g_\sigma, g_p)}{\partial (p', \sigma, p)} = \begin{pmatrix}
\frac{\partial g_{p'}}{\partial p'} & \frac{\partial g_\sigma}{\partial \sigma} & \frac{\partial g_p}{\partial p} \\
\frac{\partial g_{p'}}{\partial \sigma} & \frac{\partial g_\sigma}{\partial \sigma} & \frac{\partial g_p}{\partial \sigma} \\
\frac{\partial g_{p'}}{\partial p} & \frac{\partial g_p}{\partial p} & \frac{\partial g_p}{\partial p}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & 0 & 1 \\
-\frac{2\delta \log(p/p')}{p'} & 1 - \delta & \frac{2\delta \log(p/p')}{p'} \\
0 & 1 - \delta & 0
\end{pmatrix}
\]

Now at the fixed point, where \( p = p' \), we have for some functions \( b_1 \) of the parameters:

\[
J = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 - \delta & 0 \\
0 & b_1 & 0
\end{pmatrix}
\]

Which has eigenvalues 0, 0 and 1 - \( \delta \).

A.3.2 Leverage optimiser only

\[
J = \frac{\partial (g_{p'}, g_\sigma, g_p)}{\partial (p', \sigma, p)} = \begin{pmatrix}
\frac{\partial g_{p'}}{\partial p'} & \frac{\partial g_\sigma}{\partial \sigma} & \frac{\partial g_p}{\partial p} \\
\frac{\partial g_{p'}}{\partial \sigma} & \frac{\partial g_\sigma}{\partial \sigma} & \frac{\partial g_p}{\partial \sigma} \\
\frac{\partial g_{p'}}{\partial p} & \frac{\partial g_p}{\partial p} & \frac{\partial g_p}{\partial p}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & 0 & 0 \\
-\frac{2\delta \log(p/p')}{p'} & 1 - \delta & \frac{2\delta \log(p/p')}{p'} \\
0 & 1 - \delta & 0
\end{pmatrix}
\]

Now at the fixed point, where \( p = p' \), we have for some functions \( b_2, b_3 \) of the parameters:

\[
J = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 - \delta & 0 \\
0 & 0 & 1 - \delta \\
0 & 0 & b_2, b_3
\end{pmatrix}
\]

Which has eigenvalues 0, 0, 1 - \( \delta \), and function \( b_4 \) of the parameters, where \( 0 < b_4 < 1 \) for our range of parameters.
A.4 Stochastic simulation

Figure 13 and Figure 14 show a longer simulation for our default parameters, with deterministic and stochastic dividends respectively. For the stochastic dividends we have chosen $\epsilon = 0.05$ as in Brock and Hommes (1998).

Figure 13: Simulation with deterministic dividends

Figure 14: Simulation with stochastic dividends. All other conditions identical to Figure 13.