The Dynamics of Cultural Traits in Inherited Endogenous Social Networks

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Abstract

We study the dynamics of continuous cultural traits (as a specific type of continuous opinions) in an OLG (overlapping generation) structure and in an endogenous social network, where the network changes are inherited. Children learn their cultural trait from their parents and their social environment modelled by network. Parents want their children to adopt a cultural trait that is similar to their own and engage in the socialization process of their children by forming new links or deleting connections. Changing links from the inherited network is costly, but having many links is beneficial. We propose three ways to endogenize the process of network formation. In the first one, the network is supposed to be directed and each dynasty can either form or delete a directed link unilaterally with another dynasty. Therefore, at each period, each family faces a utility optimization problem where a trade-off between own utility losses and the improvements of child’s cultural trait. We have shown that if the cost of network changes is greater than the cost of child care, extremists will never add links, and in the case of sufficiently low cost, extremists may cut all ties with the society. In the second and third models, the network is supposed to be undirected. In the second model, we assume that after each period, a pairwise stable network (PS network for short) is reached. In this case, there always exist sufficiently small cost parameters such that the empty network is the unique PS network. In the third model, we assume that after each period, a pairwise stable network with transfers (PST network for short) is reached. We have shown the existence of the PST network for each period, however, it is not necessary to be unique. Moreover, a necessary and sufficient condition is given such that a network is PST for given $V(t)$ and $G(t)$. The convergence of cultural traits in this case is guaranteed.
Regarding the efficiency of the network, we show that there always exist sufficiently small cost parameters such that the empty network is the unique efficient network, and sufficiently large costs of child care such that the complete network is the unique efficient network. Moreover, more detailed dynamics of cultural traits are studied when the costs of network changes and benefits from integration are low, intermediate, and large, respectively.

*Keywords:* continuous cultural traits; network formation; inherited network changes; dynamics of cultural traits

*JEL-Classification:* A14; D83; D85; Z13
1. Introduction

Together with opinions and beliefs, cultural traits are part of factors that govern human behavior and are transmitted through generations. Transmission of cultural traits, such as trust, altruism, morality, risk aversion, persistence, etc., play an important role in shaping economic and social outcomes (Tabellini (2008), Bisin and Verdier (2011)). For instance, trust and risk preferences are important determinants of economic development. Hence, how the cultural traits are formed and evolved are of central interest.

The first theoretical model of cultural transmission was provided by Cavalli-Sforza and Feldman (1981) and Rindos et al. (1985), who also proposed a clear terminology that was widely-adopted by subsequent literature (Bisin and Verdier (2011)). Cavalli-Sforza and Feldman (1981) explained how the custom spread, and showed that if cultural traits were passed down solely from one generation to next generation, some demographic change could not exist, i.e., purely vertical transmission is not enough. It required also the consideration of the influence by parents’ peers and social environment, i.e., some mixture of horizontal and oblique transmission (Sober (1992)). Evidently, a large amount of empirical research has also shown that cultural traits of children are shaped by the observable cultural traits of both their parents and social environment (e.g. Dohmen et al. (2012), Tabellini (2008)). Moreover, cultural traits are observed to be persistent among generations (see, e.g., Guiso et al. (2008), Nunn and Wantchekon (2011), Algan and Cahuc (2010), Voigtländer and Voth (2012)). Thus, social networks play a central role in the process of the formation and evolution of cultural traits.

The seminal work by Bisin and Verdier (2001) provided a model of intergenerational cultural transmission for binary cultural traits and explain its global persistence theoretically, based on the assumption of imperfect empathy that parents are willing to use costly controls to increase the probability that their children adopt the same cultural trait as them. In case that a different cultural trait from their parents is adopted, children would learn the trait from a random individual among the population (i.e., oblique socialization). Imperfect empathy implies a cultural substitution effect which drives persistence of heterogeneous traits in the long run.

Buechel et al. (2014) and Panebianco (2014) extended this approach by modeling cultural traits as continuous variables and by introducing local interactions represented by a social network, considering the network as exogenously given. The role of endogenous networks is
introduced in a recent research by Hellmann and Panebianco (2018), but network changes are not inherited by supposing a fixed underlying network. The aim of this paper is to study the dynamic of continuous cultural traits in an endogenous network where network changes are inherited. Parents are able to influence their children’s networks to prevent undesirable peer effects or encourage desirable peer effects, by means of school choice, sports clubs, intervene in children’s friendships, etc. This makes central the following questions: How do parents bias children’s network optimally? How do cultural traits evolve under this presumption on optimal networks? Under which conditions do heterogeneous and homogeneous societies emerge?

These questions are closely related to whether the network is directed or not. Directed networks are good approximation to the situation where parents can guide (resp., prevent) their children to learn from some role families from public resources. For undirected networks, Pairwise stability is a simple stability concept proposed by Jackson and Wolinsky (1996) to capture the mutual consent required for forming a link between two agents, while Nash equilibrium based solutions fail to capture this point. It supposes that any individual can delete a link unilaterally, while adding a link requires the agreement of both involved individuals. Motivated by this idea of pairwise stability, some other notions of stability in network formation were proposed, e.g., pairwise Nash stability (Bloch and Jackson (2006)), pairwise stability with transfers (PST) where the transfers among individuals are allowed (Bloch and Jackson (2007)), strong stability (Dutta and Mutuswami (1997)), bilateral stability (Goyal and Vega-Redondo (2007)) and so on.

We endogenize the network formation process in the following three different frameworks: 1) the network is directed, and at each time, any adult can form or delete any link unilaterally with the other adults, by maximizing his or her own utility; 2) the network is undirected, and at each stage \( t + 1 \), a PS network is reached based on \( V(t) \) and \( G(t) \); 3) the network is undirected, and at each stage \( t + 1 \), a PST network is reached based on \( V(t) \) and \( G(t) \), i.e., the cultural traits and network of the previous stage. The first framework applies to the situations where cultural traits can be learnt by observing the behavior of others. In this case, parents can reduce or prevent their children from learning the undesirable traits. The second and third framework apply to those situations where cultural traits are influenced through bilateral contact, e.g., trust, altruism, etc. The difference is whether transfers among dynasties are allowed in the process of network formation. PS is a very standard concept
for modeling formation of undirected networks, while visible and invisible transfers are also very common when people build their connections in real life. So both notions are adopted here to study the dynamics of cultural traits.

We emphasize the role of two degrees of imperfect empathy relative to (i) cost of network changes and (ii) a desire to be integrated in the society. Moreover, we consider that a dynasty always weights more on own cultural trait than that of any other neighbor’s due to the consideration that cultural traits of children are more likely to be influenced vertically by their parents than by their neighbors (peer effects) horizontally.

This paper is structured as follows. In Section 2, we introduced the model with notions, assumptions on the networks and the utility function. How the process of network formation is endogenized (in three ways as mentioned previously) is explained in detail in Section 3, together with some remarkable results on network changes for one period. In Section 4, the dynamics (convergence, limit behavior) of these three models are discussed in detail, in both theory and simulations. In Section 5, we discuss the efficiency of networks and Section 6 concludes this paper.

2. The model

We employ the model by Buechel et al. (2014) and Panebianco (2014) for the transmission of continuous cultural traits on social networks. Consider an overlapping generations society populated by the adults of a finite set of dynasties $N = \{1, \ldots, n\}$. At the beginning of any period $t \in \mathbb{N}$, each adult has one offspring. Adults of period $t \in \mathbb{N}$ are characterized by a cultural trait $V_i(t) \in I$. As we consider continuous traits, $I \subseteq \mathbb{R}$ is assumed to be compact and convex. Following empirical evidence, children learn their cultural trait from their parents and their social environment which is determined by a social network represented by a $n \times n$ row-stochastic matrix $G = (g_{ij})_{i,j \in N}$ (i.e., $g_{ij} \geq 0$, such that $\sum_{j \in N} g_{ij} = 1$).

We assume that there is a network $G(t)$ (either directed or undirected) in each period $t \in N$. Let $N_i(t) = \{j \in N : (i, j) \in G(t)\}$ be the neighbors of dynasty $i$ in period $t$, i.e. the dynasties to whom dynasty $i$ has links in period $t$ and denote by $\eta_i(t) = |N_i(t)|$ the (out-) degree of dynasty $i$ in period $t$. 
We then get the influence network by
\[
g_{ij}(t) = \begin{cases} 
\frac{1}{n}, & \text{if } j \in N_i(t); \\
\frac{n - \eta_i(t)}{n}, & \text{if } i = j; \\
0, & \text{otherwise.}
\end{cases}
\]

In each period dynasty \( i \) can form new links or delete current links. The changes then carry over to the next generation. Obviously, \( G(t) \) is always row stochastic for any \( t \). For dynasty \( i \), parents can invest into peer socialization by adding set of links \( l_i^+ \subset N - (N_i(t) + \{i\}) \) as well as vertical socialization by deleting any set of existing links \( l_i^- \subset N_i(t) \). We denote the total investment as \( X_i = l_i^+ \cup l_i^- \). The detailed network formation process will be explained in Section 3).

Children then adopt the cultural trait according to
\[
V_i(t + 1) = \sum_{j \in N_i(t+1)} g_{ij}(t+1)V_j(t),
\]
which is formed according to the influences they are exposed to:
\[
g_{ij}(t+1) = \begin{cases} 
\frac{1}{n}, & \text{if } j \in (N_i(t) \setminus l_i^-) \cup l_i^+; \\
\frac{n - \eta_i(t) - \lambda_i^+ + \lambda_i^-}{n}, & \text{if } j = i; \\
0, & \text{otherwise.}
\end{cases}
\]

where \( \lambda_i^+ := |l_i^+| \) and \( \lambda_i^- := |l_i^-| \). At the end of any period \( t \in \mathbb{N} \), the adults die, the children become adults in period \( t + 1 \), and carry over the adopted trait into their adult period.

**Utility of each dynasty**
We assume imperfect empathy: parents want their children to adopt the same cultural trait as they carry themselves (see for a motivation e.g. Bisin and Verdier, 2010). Moreover, as both parental socialization and network changes are costly, we assume the following functional form of utility
\[
U_i(t + 1) = -[V_i(t + 1) - V_i(t)]^2 - c_i^A(\lambda_i^+ + \lambda_i^-)^2 - c_i^\eta(n - \eta_i(t + 1)),
\]
where \( c_i^A, c_i^\eta > 0 \) for all \( i \in \mathbb{N} \).

The first term \(-[V_i(t + 1) - V_i(t)]^2\) is the intergenerational utility component which is decreasing in the distance between parent’s and child’s trait and therefore reflects imperfect
empathy. The remaining term is composed of the cost of network intervention and the cost of parental socialization. Cost of network intervention depends on the number of altered links between $G_i(t + 1)$ and the previous original influence $G_i(t)$ weighted by the dynasty dependent cost factor $c^\Delta$. Cost of parental socialization is created by the amount of time parents spend with the children $\frac{n - n_i(t + 1)}{n}$ weighted with a cost term $nc^\eta$. This cost term can also be interpreted as a (positive) benefit term from interaction with others.

3. Endogenize network formation ($G(t) \rightarrow G(t + 1)$)

In this section, we propose three ways to endogenize the network formation process, based on whether the network is directed or not and stability of the network dynamics. In Section 3.1, the network is supposed to be directed and thus dynasty can form or delete any link unilaterally. In Section 3.3 and Section 3.2, the network is supposed to be undirected. We endogenize the undirected network formation by adopting the notions of PST and PS networks in Section 3.3 and Section 3.2, respectively.

3.1. Directed network with optimal network changes

3.1.1. Optimization problem of purposeful socialization

Suppose that $G(t)$ is directed and thus dynasties face a trade-off between own utility losses (due to costs of child care and network changes) and eventual improvements in locations of child’s trait relative to peak. Thus the optimization problem of adult $i$ in period $t + 1$ is:

$$\max_{\mathcal{I}^+_{i}, \mathcal{I}^-_{i}} - (V_i(t + 1) - V_i(t))^2 - c^\Delta(\lambda^+_i + \lambda^-_i)^2 - c^\eta(n - \eta_i(t) - \lambda^+_i + \lambda^-_i)^2 - c^\eta(n - \eta_i(t) - \lambda^+_i + \lambda^-_i)^2$$

s.t. $V_i(t + 1) = \sum_{j \in \mathcal{N}} g_{ij}(t + 1) V_j(t)$

$$= V_i(t) + \frac{1}{n} \sum_{j \in (\mathcal{N}_i(t) \setminus \mathcal{I}^-_{i}) \cup \mathcal{I}^+_{i}} (V_j(t) - V_i(t)).$$

3.1.2. Optimal network changes in each period

Consider the change of network for dynasty $i \in \mathcal{N}$. Adult $i$’s utility then changes according to

$$[U_i(g(t + 1)) - c^\Delta(g(t + 1) \mid g(t)) - c^\eta(g(t + 1))] - [U_i(g(t)) - c^\eta(g(t))]$$

$$= -\frac{1}{n^2} \left[ d^i_{\mathcal{I}^+_{i}} - d^i_{\mathcal{I}^-_{i}} \right] \left[ d^i_{(\mathcal{N}_i(t) \setminus \mathcal{I}^-_{i}) \cup \mathcal{I}^+_{i}} - (\lambda^+_i - \lambda^-_i)c^\eta - (\lambda^+_i + \lambda^-_i)^2 c^\Delta \right],$$

where for $L \subseteq \mathcal{N} \setminus \{i\}$ we have set $d^L_{i} := \sum_{j \in L} (V_i(t) - V_j(t))$, and the set operations union and subtraction are denoted as $+$ and $-$ among sets.
Denote by $V_i^<(t) := \{j \in N : V_j(t) < V_i(t)\}$ and by $V_i^>(t) := \{j \in N : V_j(t) > V_i(t)\}$. A necessary condition for optimal links is given by the following Proposition.

**Proposition 1.** Suppose $c^n \leq c^A$. The necessary conditions for a set of links $L_i^+$ added to network $g(t)$ and a set of links $L_i^-$ deleted from network $g(t)$ by player $i \in N$ to be optimal are that:

- **A)** for all $j \in L_i^+ \cup V_i^<(t)$ it holds that $d_i^{N+L_i^+-L_i^-j}(t) < 0$,
- **B)** for all $j \in L_i^+ \cup V_i^>(t)$ it holds that $d_i^{N+L_i^+-L_i^-j}(t) > 0$,
- **C)** for all $j \in L_i^- \cup V_i^<(t)$ it holds that $d_i^{N+L_i^-+L_i^-j}(t) > 0$,
- **D)** for all $j \in L_i^- \cup V_i^>(t)$ it holds that $d_i^{N+L_i^-+L_i^-j}(t) < 0$.

**Proof.** We only show the first point, the remainder is completely analogous. Suppose that contrarily to the assertion we have $j \in L_i^+ \cup V_i^<(t)$ but $d_i^{N+L_i^+-L_i^-j}(t) > 0$. Since $j \in V_i^<(t)$ we have $d_i^j(t) = (V_i(t) - V_j(t)) > 0$. We get that

$$d_i^{N+L_i^+-L_i^-j}(t) = \sum_{k \in N+L_i^+-L_i^-} (V_i(t) - V_k(t))$$

$$= \sum_{k \in N+L_i^+-L_i^-} (V_i(t) - V_k(t)) + (V_i(t) - V_k(t))$$

$$= d_i^{N+L_i^+-L_i^-j}(t) + d_i^j(t) > 0.$$

For the marginal utility of the link $ij$ we then get:

$$u_i(g(t) + L_i^+ - L_i^-) - u_i(g(t) + L_i^+ - L_i^- - j)$$

$$= -\frac{1}{n^2} \left( d_i^j(t) \left( d_i^{N_i(t)+L_i^+-L_i^-j}(t) + d_i^{N_i(t)+L_i^+-L_i^-j}(t) \right) + c^n - c^A(2(l_i^+ + l_i^-) + 1) \right)$$

$$< c^n - c^A(2(l_i^+ + l_i^-) + 1) \leq 0.$$  

The last inequality is due to the fact that $c^n \leq c^A$.  

Note that condition C and condition D are mutually exclusive. Thus either $L_i^-(t) \subset V_i^<(t)$ or $L_i^-(t) \subset V_i^>(t)$. From equation (1), we get that

$$V_i(t+1) = V_i(t) - \frac{1}{n} d_i^{N_i(t+1)}.$$  

Then it is obvious that if condition C holds, $V_i(t+1) < V_i(t)$; if condition D (thus D’) holds, $V_i(t+1) > V_i(t)$. It means that if agent cuts a link to an agent with greater cultural trait in period $t$, her cultural trait in period $t+1$ will become greater; vice versa.
Single link changes. The following corollary shows that link changes are only optimal if these countervail the peer influence.

**Corollary 1.** Suppose \( c^1 \leq c^\Delta \) and let \( |l_i^+ \cup l_i^-| \leq 1 \) for some \( i \in N \). Then this can be optimal only if:

(i) If \( V_{1}(t) < \frac{1}{|N(t)|} \sum_{j \in N_i(t)} V_{j}(t) \), then \( l_i^+ \subset V_{i}^{>} (t) \) and \( l_i^- \subset V_{i}^{<} (t) \).

(ii) If \( V_{1}(t) > \frac{1}{|N(t)|} \sum_{j \in N_i(t)} V_{j}(t) \), then \( l_i^- \subset V_{i}^{=} (t) \) and \( l_i^+ \subset V_{i}^{>} (t) \).

(iii) If \( V_{1}(t) = \frac{1}{|N(t)|} \sum_{j \in N_i(t)} V_{j}(t) \), then \( l_i^- = l_i^+ = \emptyset \).

**Extremists’ behavior.** We call the adults who hold the lowest or the greatest cultural traits as the extremists. The following corollaries show that extremists never add links and in the case of sufficiently low cost, extremists may cut all ties with the society.

**Corollary 2.** Suppose \( c^1 \leq c^\Delta \) and let \( i \in \text{arg min}_{k \in N} V_k(t) \) and \( j \in \text{arg max}_{k \in N} V_k(t) \). Then, \( l_i^+ = l_j^+ = \emptyset \).

**Proof.** For all \( i \in \text{min}_{k \in N} V_k(t) \) it is \( d_{N}(i) \leq 0 \) for all \( N_i \subset N \setminus \{i\} \) and \( V_{i}^{=}(t) = \emptyset \). Thus the first two conditions of Proposition 1 cannot be satisfied implying that \( L_i^+ = \emptyset \).

Similarly for all \( j \in \text{max}_{k \in N} V_k(t) \) it is \( d_{N}(j) \geq 0 \) for all \( N_j \subset N \setminus \{j\} \) and \( V_{j}^{>}(t) = \emptyset \). Thus, again, the first two conditions of Proposition 1 cannot be satisfied implying that \( L_j^+ = \emptyset \).

**Corollary 3.** Let \( i \in \text{arg min}_{k \in N} V_k(t) \) and \( j \in \text{arg max}_{k \in N} V_k(t) \). There exists \( \epsilon > 0 \) such that for all \( 0 \leq c^1 \leq c^\Delta < \epsilon \), we get \( l_i^- = V_{i}^{=} \cap N_i(t) \) and \( l_j^- = V_{j}^{=} \cap N_j(t) \).

The following example illustrates the network changes in one period.

**Example 1.** Consider 10 dynasties situated in an initial network given by Figure 1 with its incidence matrix being

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 9 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

and the costs of child care and network changes are \( c^1 = 0 \) and \( c^\Delta = 0.0045 \), respectively. The initial state of cultural traits is \( V(0) = (1, 2, 3, 3, 4, 4, 5, 7, 9, 11) \). To illustrate the differences of network changes between extremists and integrated families, take \( n = 1, 7, 10 \) as an example. See Figure 2 for the network structure, restricted to dynasties 1, 7 and 10. By
solving the corresponding optimization problems, dynasty 1 will delete links with 7 and 9, 
dynasty 10 will delete links with 2 and 3, and dynasty 7 would rather add links to dynasty 10 
(see Figure 3). The resulting network at time 1 is shown in Figure 4 with its incidence matrix to be

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

Figure 1: Initial network.

3.2. Undirected network with PS (Pairwise Stable) networks

Suppose that \(G(t)\) is undirected. The PS network is defined as follows.

**Definition 1 (Pairwise Stable).** For all \(V(t), g(t)\), a network \(g^*\) is Pairwise Stable (PS for short) if

(i) \(\forall ij \in g^*\),
\[
u_i(g^* - ij|g(t), V(t)) - u_i(g^*|g(t), V(t)) \leq 0,
\]
and \(u_j(g^* - ij|g(t), V(t)) - u_j(g^*|g(t), V(t)) \leq 0;\)

\(^1u_i(g^* - ij|g(t), V(t))\) refers to the utility of agent \(i\) of deleting the link \(ij\) from \(g^*\) given the current network \(g(t)\) and current cultural trait vector \(V(t)\). It is analogous for \(u_i(g^* + ij|g(t), V(t))\).
Figure 2: Initial network, restricted to dynasties 1,7 and 10.

Figure 3: Network changes of 1,7 and 10, with deleted links colored in blue and added links colored in green.

(ii) \( \forall ij \notin g^* \), 
\[ u_i(g^* + ij|g(t), V(t)) > u_i(g^*|g(t), V(t)) \implies u_j(g^* + ij|g(t), V(t)) \leq u_j(g^*|g(t), V(t)). \]

Pairwise stable network requires that no adult wants to delete a link unilaterally and no two unconnected adults both want to form a link, considering one link at a time. It requires the mutual agreement to form a link between two adults.

The following proposition gives the sufficient condition such that the empty network is Pairwise Stable.

**Proposition 2.** For any \( V(t) \) and \( g(t) \), a sufficient condition such that the empty network is Pairwise Stable for given \( V(t), g(t) \) is that \( \frac{c_i}{c_j} \leq \min_{i \in N} \{2x_i + 1\} \), where \( x_i \) and \( x_j \) are the numbers of links agent \( i \) and \( j \) have in \( g(t) \), respectively.

**Proof.** Suppose that in \( g(t) \), agent \( i \) has \( x_i \) links, \( \forall i \in N \). Then

\[
mu_i(g^0 + ij, ij|V(t), g(t))
\]
Thus \( \frac{c^n}{c^\Delta} \leq 2x_i + 1 \implies m u_i (g^\emptyset + ij, ij|V(t), g(t)) \leq 0. \)

So

\[
\frac{c^n}{c^\Delta} \leq \min_{i \in N} \{2x_i + 1\}
\]

\[
\implies m u_i (g^\emptyset + ij, ij|V(t), g(t)) \leq 0, \forall i \in N
\]

\[
\implies g^\emptyset \text{ is PS.}
\]

**Remark 1.** When \( c^n = c^\Delta = 0 \), \( g^\emptyset \) is PS.

Proposition 2 says that if \( \frac{c^n}{c^\Delta} \) is sufficiently small, the empty network will be pairwise stable. Furthermore, if the two cost parameters are sufficiently small, the empty network will be the unique pairwise stable network.

**Proposition 3.** There always exist sufficiently small \( c^n \) and \( c^\Delta \) such that the empty network is the unique PS network for given \( g(t) \) and \( V(t) \).

**Proof.** Given \( g(t) \) and \( V(t) \), Proposition 2 guarantees the existence of \( c^\Delta \) and \( c^n \) such that the empty network is PS. It suffices to prove that \( \forall g \neq g^\emptyset, \exists c^\Delta > 0 \) and \( c^n > 0 \) such that \( g \) is not PS.

We will show that for any nonempty network is not PS. Fix any \( g \neq g^\emptyset \), among all the agents having at least one link in \( g \), we can always find the agent holding either the largest or the smallest cultural trait, say \( i \), and \( i \) wants to delete the links. Suppose w.l.o.g. \( i \) is connected to those holding larger culture traits, i.e., \( V_j > V_i, \forall j \in N_i \), where \( N_i \) is the neighborhood of agent \( i \) in \( g \). Denote \( y_i \) as the number of links that agent \( i \) need to change from \( g(t) \) to \( g \). Depending on whether \( ij \in G(t) \) or not, the number of links that agent \( i \) need to change from \( G(t) \) to \( g - ij \) can be computed as \( y_i + 1 \) (for the case of \( ij \in G(t) \)
or $y_i - 1$ (for the case of $ij \notin G(t)$). Denote $N_i$ as the neighborhood of agent $i$ in $g$. The marginal utility of deleting any link $ij$ from $g$ for agent $i$ is

$$mu_i(g, ij | V(t), G(t)) = \frac{1}{n^2}[(d_i^N - (j))^2 - (d_i^N)^2] + c^\Delta (1 + 2y_i) + c^\eta.$$ 

When

$$c^\Delta (1 + 2y_i) + c^\eta < \frac{1}{n^2}[(d_i^N - (j))^2 - (d_i^N)^2],$$

$$mu_i(g, ij | V(t), G(t)) < 0,$$ i.e., agent $i$ would like to delete the link $ij$. Since $\forall j \in N_i$, $V_j > V_i$, $(d_i^N - (j))^2 - (d_i^N)^2 > 0$. Thus we can always find small enough $c^\Delta$ and $c^\eta$ such that inequality 4 holds, so $g$ is not PS.

Above all, $g^0$ is the unique PS network.

3.3. Undirected network with PST (Pairwise Stable with Transfers) networks

In this section and the next section, we suppose that $G(t)$ is undirected. The PST network is defined as follows.

**Definition 2 (Pairwise Stable with Transfers).** For all $V(t), g(t)$, a network $g^*$ is Pairwise Stable with Transfers (PST) if

(i) $\forall ij \in g^*$,
$$U_i(g^* | g(t), V(t)) + u_j(g^* | g(t), V(t)) \geq U_i(g^* - ij | g(t), V(t)) + u_j(g^* - ij | g(t), V(t));$$

(ii) $\forall ij \notin g^*$,
$$U_i(g^* | g(t), V(t)) + u_j(g^* | g(t), V(t)) \geq U_i(g^* + ij | g(t), V(t)) + u_j(g^* + ij | g(t), V(t)).$$

Pairwise Stable network with Transfers requires that no pair of agents can jointly benefit by forming or deleting a link. In other words, it allows the utility exchange between two agents.

3.3.1. Existence of PST networks at each period

Jackson and Watts (2002) showed that there exist at least one pairwise stable network or closed cycle of networks. Following the same reasoning, it also holds for PST networks.

**Lemma 1.** For any network society, there exists at least one PST network or a closed cycle of networks.

**Theorem 1.** For all $V(t)$ and $g(t)$, there exist at least one PST network.

**Proof.** We show here there is no closed cycle. For any given $g(t)$, assume there is a closed cycle $g^1, g^2, \ldots, g^m$, where $g^1 = g^m = g(t)$. Note here that $u_i(g) = u_i(g')$ if $N_i(g) = N_i(g')$. Therefore the sum of the utilities is strictly increasing along the improving path, i.e., $\sum_{i=1}^{n} u_i(g^k) < \sum_{i=1}^{n} u_i(g^{k+1}), \forall k \in \{1, 2, \ldots, m - 1\}$. Thus $\sum_{i=1}^{n} u_i(g^1) < \sum_{i=1}^{n} u_i(g^m)$ leading to a contradiction to $\sum_{i=1}^{n} u_i(g^1) = \sum_{i=1}^{n} u_i(g^m) = \sum_{i=1}^{n} u_i(g(t)))$. So there is no closed improving cycle but at least one pairwise stable network with transfers by Lemma 1. 

\[\square\]
3.3.2. Network changes in one period

We suppose that in each period, a PST network $G(t+1)$ is reached given $V(t)$ and $G(t)^2$. The following example illustrates the network changes in one period.

**Example 2.** Consider 10 dynasties situated in an initial network given by Figure 5 with its incidence matrix being

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and the costs of child care and network changes are $c_\eta = 0.01$ and $c_\Delta = 0.002$, respectively. The initial state of cultural traits is $V(0) = (1, 2, 3, 3, 4, 4, 5, 7, 9, 11)$. To illustrate the differences of network changes between extremists and integrated families, take $n = 1, 3, 10$ as an example. See Figure 6 for the network structure, restricted to dynasties 1, 6 and 10. By solving the corresponding optimization problems, dynasty 1 will delete links with 6, 7 and 9, dynasty 10 will delete links with 6, and dynasty 3 would delete link with 9, and at the same time add link with 4 (see Figure 7). The resulting network at time 1 is shown in Figure 8 with its incidence matrix to be

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In case that a PST network is reached in each period, extremists tend to cut links and intermediate agents tend to add links.

\footnote{In case of the existence of multiple PST networks, a random PST network will be reached.}
3.3.3. Necessary and sufficient conditions for a network to be PST

Given $V(t)$ and $G(t)$, we define the sum of marginal utilities of $i$ and $j$ for adding a link to $g^*$ or cutting the link $ij$ from $g^*$ as follows.

**Definition 3 (Sum of marginal utilities of $i$ and $j$).**

The sum of marginal utilities of $i$ and $j$ for adding a link to $g^*$ is defined as

$$mu_{i+j}(g^* + ij|g(t), V(t)) := u_i(g^* + ij|g(t), V(t)) - u_i(g^*|g(t), V(t)) + u_j(g^* + ij|g(t), V(t)) - u_j(g^*|g(t), V(t)).$$

The sum of marginal utilities of $i$ and $j$ for cutting the link $ij$ from $g^*$ is defined as

$$mu_{i+j}(g^*, ij|g(t), V(t)) := u_i(g^*|g(t), V(t)) - u_i(g^* - ij|g(t), V(t)) + u_j(g^*|g(t), V(t)) - u_j(g^* - ij|g(t), V(t)).$$
Figure 7: Network changes of 1, 3 and 10, with deleted links colored in blue and added links colored in green.

Figure 8: Resulting network at time 1

The following proposition provides the necessary and sufficient condition such that the empty network is PST network.

**Proposition 4.** $\forall i, j \in N$, suppose that in $g(t)$, $i$ has $x_i$ links and $j$ has $x_j$ links. For any $V(t), g(t)$, the necessary and sufficient condition such that the empty network is PST network is that

$$-rac{(v_i - v_j)^2}{n^2} \leq c^\Delta - c^n - c^\Delta (x_i + x_j), \forall ij \in g(t), \text{ and}$$

$$-rac{(v_i - v_j)^2}{n^2} \leq c^\Delta - c^n + c^\Delta (x_i + x_j), \forall ij \notin g(t)$$

*Note that when $c^n = c^\Delta = 0$, $g^0$ is always a PST network.*

*Proof.* The sum of the marginal utilities of adding any link $ij$ to $g^0$ is
\[ u_i(g^0 + ij|g(t), V(t)) + u_j(g^0 + ij|g(t), V(t)) - u_i(g^0|g(t), V(t)) - u_j(g^0|g(t), V(t)) = -\frac{2(v_i - v_j)^2}{n^2} - c^\Delta[(x_i - 1)^2 - x_i^2 + (x_j - 1)^2 - x_j^2] - 2c^\eta(n - 1 - n) \]

\[ = -\frac{2(v_i - v_j)^2}{n^2} - 2c^\Delta + 2c^\eta + 2c^\Delta(x_i + x_j), \text{ if } ij \in g(t); \]

\[ = -\frac{2(v_i - v_j)^2}{n^2} - 2c^\Delta + 2c^\eta - 2c^\Delta(x_i + x_j), \text{ if } ij \notin g(t). \]

When \( ij \in g(t), \) a sufficient condition such that \((mu_i + mu_j)(g^0 + ij, ij) < 0\) is that \(2c^\Delta - 2c^\eta - 2c^\Delta(x_i + x_j) > 0,\) i.e. \(\frac{c^\eta}{c^\Delta} < 1 - (x_i + x_j);\) (So there could be the case that the empty network is not PST).

When \( ij \notin g(t), \) a sufficient condition such that \((mu_i + mu_j)(g^0 + ij, ij) < 0\) is that \(2c^\Delta - 2c^\eta + 2c^\Delta(x_i + x_j) > 0,\) i.e. \(\frac{c^\eta}{c^\Delta} < 1 + (x_i + x_j).\)

\(g^0\) is PST \(\iff (mu_i + mu_j)(g^0 + ij, ij) \leq 0\)
\[ \iff -\frac{(v_i - v_j)^2}{n^2} \leq c^\Delta - c^\eta - c^\Delta(x_i + x_j), \forall ij \in g(t) \]
\[ \text{and} \quad -\frac{(v_i - v_j)^2}{n^2} \leq c^\Delta - c^\eta + c^\Delta(x_i + x_j), \forall ij \notin g(t). \]

More generally, the following proposition gives the necessary and sufficient condition such that any \(g^*\) is PST network.

**Proposition 5.** For any \(V(t), g(t), g^*,\) suppose that \(\forall i \in N, i \text{ has } x_i \text{ links changed from } g(t) \text{ to } g^*.\) The neighbourhood of player \(i\) in networks \(g(t), g^*, g^* + ij\) are denoted as \(N_i, N_i^*, N_i^* + \{j\},\) with \(|N_i| = \eta_i, |N_i^*| = \eta_i^*\) and \(|N_i^* + \{j\}| = \eta_i^* + 1\) respectively. For any \(V(t), g(t),\) the necessary and sufficient condition such that \(g^*\) is PST network is that

\[ \forall ij \notin g^*, \frac{1}{n^2}[\langle V_i - V_j \rangle (d_i^{N_i^*} - d_j^{N_j^*} + V_i - V_j)] + c^\Delta(x_i + x_j - 1) + c^\eta \leq 0, \text{ if } ij \in g(t), \]
\[ \frac{1}{n^2}[\langle V_i - V_j \rangle (d_i^{N_i^*} - d_j^{N_j^*} + V_i - V_j)] - c^\Delta(x_i + x_j - 1) + c^\eta \leq 0, \text{ if } ij \notin g(t); \]
\[ \forall ij \in g^*, \frac{1}{n^2}[\langle V_i - V_j \rangle (d_i^{N_i^*} - d_j^{N_j^*} + V_i - V_j)] - c^\Delta(x_i + x_j - 1) + c^\eta \leq 0, \text{ if } ij \in g(t), \]
\[ \frac{1}{n^2}[\langle V_i - V_j \rangle (d_i^{N_i^*} - d_j^{N_j^*} + V_i - V_j)] + c^\Delta(x_i + x_j - 1) + c^\eta \leq 0, \text{ if } ij \notin g(t). \]

**Proof.**

\(g^*\) is a PST network given \(V(t)\) and \(g(t)\)
\[ \iff mu_{i+j}(g^* + ij, g^*|V(t), g(t)) \leq 0, \forall ij \notin g^*; \]
\[ \text{and} \quad mu_{i+j}(g^*g^* - ij|V(t), g(t)) \geq 0, \forall ij \in g^*. \]

We distinguish the following cases:
• $ij \notin g^*$
  - $ij \in g(t)$

  $$\mu_{i+j}(g^* + ij, g^* | V(t), g(t))$$
  $$= - \left( \frac{1}{n} d_i^{N_i^*} + (j) \right)^2 - c^\Delta (x_i + 1)^2 - c^\delta (n - \eta_i^* - 1)$$
  $$- \left( \frac{1}{n} d_j^{N_j^*} + (i) \right)^2 - c^\Delta (x_j + 1)^2 - c^\delta (n - \eta_j^* - 1)$$
  $$+ \left( \frac{1}{n} d_i^{N_i^*} \right)^2 - c^\Delta (x_i)^2 - c^\delta (n - \eta_i^*)$$
  $$+ \left( \frac{1}{n} d_j^{N_j^*} \right)^2 - c^\Delta (x_j)^2 - c^\delta (n - \eta_j^*)$$
  $$= \frac{2}{n^2} [(V_i - V_j)(d_i^{N_i^*} - d_j^{N_j^*} + V_i - V_j)] + 2c^\Delta (x_i + x_j - 1) + 2c^\delta;$$

  - $ij \notin g(t)$

  $$\mu_{i+j}(g^* + ij, g^* | V(t), g(t))$$
  $$= - \left( \frac{1}{n} d_i^{N_i^*} + (j) \right)^2 - c^\Delta (x_i + 1)^2 - c^\delta (n - \eta_i^* - 1)$$
  $$- \left( \frac{1}{n} d_j^{N_j^*} + (i) \right)^2 - c^\Delta (x_j + 1)^2 - c^\delta (n - \eta_j^* - 1)$$
  $$+ \left( \frac{1}{n} d_i^{N_i^*} \right)^2 - c^\Delta (x_i)^2 - c^\delta (n - \eta_i^*)$$
  $$+ \left( \frac{1}{n} d_j^{N_j^*} \right)^2 - c^\Delta (x_j)^2 - c^\delta (n - \eta_j^*)$$
  $$= \frac{2}{n^2} [(V_i - V_j)(d_i^{N_i^*} - d_j^{N_j^*} + V_i - V_j)] - 2c^\Delta (x_i + x_j - 1) + 2c^\delta;$$

• $ij \in g^*$
  - $ij \in g(t)$

  $$- \mu_{i+j}(g^*, ij | V(t), g(t))$$
  $$= - \left( \frac{1}{n} d_i^{N_i^*} - (j) \right)^2 - c^\Delta (x_i + 1)^2 - c^\delta (n - \eta_i^* - 1)$$
  $$- \left( \frac{1}{n} d_j^{N_j^*} - (i) \right)^2 - c^\Delta (x_j + 1)^2 - c^\delta (n - \eta_j^* - 1)$$
  $$+ \left( \frac{1}{n} d_i^{N_i^*} \right)^2 - c^\Delta (x_i)^2 - c^\delta (n - \eta_i^*)$$
  $$+ \left( \frac{1}{n} d_j^{N_j^*} \right)^2 - c^\Delta (x_j)^2 - c^\delta (n - \eta_j^*)$$
  $$= \frac{2}{n^2} [(V_i - V_j)(d_i^{N_i^*} - d_j^{N_j^*} + V_i - V_j)] - 2c^\Delta (x_i + x_j - 1) + 2c^\delta;$$

  - $ij \notin g(t)$

  $$- \mu_{i+j}(g^*, ij | V(t), g(t))$$
  $$= - \left( \frac{1}{n} d_i^{N_i^*} - (j) \right)^2 - c^\Delta (x_i - 1)^2 - c^\delta (n - \eta_i^* - 1)$$
  $$- \left( \frac{1}{n} d_j^{N_j^*} - (i) \right)^2 - c^\Delta (x_j - 1)^2 - c^\delta (n - \eta_j^* - 1)$$
\[
\begin{align*}
&+ \left( -\frac{1}{n} d_i^N \right)^2 - c^\Delta (x_i)^2 - c^\eta (n - \eta_i^*) \\
&+ \left( -\frac{1}{n} d_j^N \right)^2 - c^\Delta (x_j)^2 - c^\eta (n - \eta_j^*) \\
&= \frac{2}{n^2} (V_i - V_j) (d_j^N - d_i^N + V_i - V_j)] + 2c^\Delta (x_i + x_j - 1) + 2c^\eta.
\end{align*}
\]

4. Dynamics

For simplicity, we denote the dynamic model of cultural traits in directed networks (described in Section 3.1), in undirected PST networks (described in Section 3.3) and in undirected PS networks (described in Section 3.2) as the DN model, the UPST model and the UPS model, respectively.

4.1. The DN model

The optimization Problem for adult \( i \in N \) at time \( t \) is:

\[
\max_{l_i^+, l_i^-} -(V_i(t + 1) - V_i(t))^2 - c^\Delta (\lambda_i^+ + \lambda_i^-)^2 - c^\eta (n - \eta(t) - \lambda_i^+ + \lambda_i^-)
\]

\[
= \max_{l_i^+, l_i^-} -\frac{1}{n^2} (d_i^N(t+1))^2 - c^\Delta (\lambda_i^+ + \lambda_i^-)^2 - c^\eta (n - \eta(t) - \lambda_i^+ + \lambda_i^-).
\]

A steady state is such that either all traits are homogenous or the extremists are disconnected from all others.

**Definition 4.** A state \( V(t) \) is called a steady state if for all \( t' > t \), \( V_i(t') = V_i(t) \), for all \( i \in N \).

4.1.1. General results

Assume that each dynasty applies optimal network changes in each period. If at some time \( t \), a dynasty \( i \) deleted all the links with the others, i.e., \( N_i(t) = \emptyset \), then the dynasty will never add links.

**Proposition 6.** Assume that \( c^\eta < c^\Delta \). \( \forall i \in N \), if \( \exists t \), such that \( N_i(t) = \emptyset \), then \( N_i(t') = \emptyset, \forall t' > t \). As a result, \( V_i(t') = V_i(t) \), \( \forall t' > t \).

**Proof.** The utility of adding any number of links is decreasing for every dynasty since the intergenerational utility would be non-increasing by linking to other dynasties and the sum of the cost of network intervention and the cost of parental socialization would be decreasing due to the assumption that \( c^\eta < c^\Delta \). \( \square \)

The following proposition gives the necessary conditions for agents with only one link to add one more link, or to add one more link and to delete the existing link at the same time, respectively.
Proposition 7. For any \( n \in \mathbb{N} \), assume adult \( i \) has one link to adult \( j \) at time \( t \), i.e., \( N_i(t) = \{j\} \). Then \( N_i(t+1) \neq \{k\} \) if
\[
c^\Delta > \frac{1}{4n^2}((d_i^j(t))^2 - (d_i^k(t))^2);
\]
\( N_i(t+1) \neq \{j,k\} \) if
\[
c^\Delta - c^n > -\frac{1}{n^2}((d_i^{j,k}(t))^2 - (d_i^j(t))^2).
\]

Proof.
\[
U_i(N_i(t+1) = \{j\})|N_i(t) = \{j\} = -\frac{1}{n^2}((d_i^j(t))^2 - (n-1)c^n).
\]
\[
U_i(N_i(t+1) = \{j,k\})|N_i(t) = \{j\} = -\frac{1}{n^2}((d_i^{j,k}(t))^2) - c^n - (n-2)c^\Delta.
\]
\[
U_i(N_i(t+1) = \{k\})|N_i(t) = \{j\} = -\frac{1}{n^2}((d_i^k(t))^2) - (n-1)c^n - 4c^\Delta.
\]

Adult \( i \) will keep the link with adult \( j \) and form a link with adult \( k \) only if \( U_i(N_i(t+1) = \{j,k\})|N_i(t) = \{j\} > U_i(N_i(t+1) = \{j\})|N_i(t) = \{j\} \) which leads to \( c^\Delta - c^n < -\frac{1}{n^2}((d_i^{j,k}(t))^2) - (d_i^j(t))^2) \).

Adult \( i \) will dellete the link with adult \( j \) and form a link with adult \( k \) only if \( U_i(N_i(t+1) = \{k\})|N_i(t) = \{j\} > U_i(N_i(t+1) = \{j\})|N_i(t) = \{j\} \) which leads to \( c^\Delta < \frac{1}{4n^2}((d_i^j(t))^2 - (d_i^k(t))^2) \).

\[
Proposition 7 tells that adult would not form other links if the current cultural distance with the neighbourhood is sufficiently small.

Corollary 4. Assume \( n = 3 \) and adult \( i \) has one link to adult \( j \) at time \( t \), i.e., \( N_i(t) = \{j\} \). Then \( N_i(t+1) = \{j,k\} \) only if
\[
c^\Delta - c^n < -\frac{1}{9}((V_j(t) + V_k(t) - 2V_i(t))^2) - (V_j(t) - V_i(t))^2
\]
which also can be written as
\[
c^\Delta - c^n < -\frac{1}{9}((d_i^{j,k}(t))^2 - (d_i^j(t))^2);
\]
\( N_i(t+1) = \{k\} \) only if
\[
c^\Delta < \frac{1}{36}((d_i^j(t))^2 - (d_i^k(t))^2).
\]
This is equivalent to say that \( N_i(t+1) \neq \{j,k\} \) if
\[
c^\Delta - c^n > -\frac{1}{9}((d_i^{j,k}(t))^2 - (d_i^j(t))^2);
\]
\( N_i(t+1) \neq \{k\} \) if
\[
c^\Delta > \frac{1}{36}((d_i^j(t))^2 - (d_i^k(t))^2).
\]
Then, for any $\epsilon > 0$, if there exists $\epsilon$ such that the traits of the whole society converge to that of an extremist subgroup.

4.1.3. Intermediate costs

Extremists disconnect and it leads to a long term heterogeneity.

Example 3. Consider the same initial network structure and cultural traits with Example

Proposition 8. Let $c^\eta < c^\Delta$ small enough such that

$$c^\Delta < \min_{i \in N} \frac{\min(V_i(0) - V_j(0))^2}{n^2((n - 1)^2 + \eta_i(0) + 1)}.$$ 

Then, for any $\epsilon > 0$ there exists $t_0 \in \mathbb{N}$ such that for $\underline{V}(t) := \min_{k \in N} V_k(t)$ and $\bar{V}(t) := \max_{k \in N} V_k(t)$ we get that there exists $\underline{V}, \bar{V} \in \mathbb{R}$ such that $|\underline{V} - \underline{V}(t)| < \epsilon$ and $|\bar{V} - \bar{V}(t)| < \epsilon$ for all $t \geq t_0$ and $\bar{V} - \underline{V} > 0$.

Proof. We will show that if $c^\Delta < \frac{\min (V_i(0) - V_j(0))^2}{n^2((n - 1)^2 + \eta_i(0) + 1)}$, the optimal network at time 1 will be the empty network and stay empty after time 1.

For any $i \in N$, if he deletes all his links to others, his utility of deleting all links will be $u_i^\emptyset(1) = -c^\Delta(n - 1)^2 - c^\eta(\eta_i + 1)$. The utility of any other network changes $u_i^{g'}(1) < -\frac{1}{n^2} \min_{j \in N} (V_i(0) - V_j(0))^2$, for any $g' \neq g^\emptyset$. Thus $u_i^{g^\emptyset}(1) > u_i^{g^\emptyset}(1), \forall g' \neq g^\emptyset$. This holds for any $i \in N$, so the network will be $g^\emptyset$ after time 1.

Example 3. Consider the same initial network structure and cultural traits with Example 1 and $c^\eta = 0.001$, $c^\Delta = 0.003$. The dynamic of cultural traits is shown in Figure 9. The extremists disconnect and it leads to a long term heterogeneity.

4.1.2. Small costs

Assume the initial state is heterogeneous, i.e., there exist $i$ and $j$ such that $V_i(0) \neq V_j(0)$.

If both costs of changing the network and cost of child care are small (relative to the degree of imperfect empathy), then the extremists (possibly groups) will disconnect and there will be long term heterogeneity.

4.1.3. Intermediate costs

We can almost always find (intermediate) cost values (relative to degree of imperfect empathy) such that the traits of the whole society converge to that of an extremist subgroup.
Figure 9: Dynamic of Cultural Traits for $c^\eta = 0.001$, $c^\Delta = 0.003$.

**Conjecture 1.** Suppose the initial traits are randomly distributed according to the uniform distribution on some interval $I \subset \mathbb{R}$ and suppose the initial network is a Bernoulli random network. Then, for almost all initial cultural traits $V(0)$ and for all initial networks $g(0)$, there exists costs $0 \leq c^\eta \leq c^\Delta$ and a (strict) subset of players $E \subset N$ such that for all $\epsilon > 0$ there exists a $t_0 \in \mathbb{N}$:

$$g_{ij}(t) = 0 \forall i \in E, j \in N \setminus E \quad \text{and} \quad |V_i(t) - V_j(t)| < \epsilon.$$

**Example 4.** Consider the same initial network structure and cultural traits with Example 1 and $c^\eta = 0.009$, $c^\Delta = 0.023$. The dynamic of cultural traits is shown in Figure 10. The traits of the whole society converge to the lowest extremist.

**4.1.4. Large costs**

Large costs of either child care or cost of network change (relative to the degree of imperfect empathy) imply convergence to a homogenous society.

**Proposition 9.** Let $c^\eta$ or $c^\Delta$ be large enough and suppose for all $i, j \in N$ there exists a directed path from $i$ to $j$ in $g(0)$. Then, for all $\epsilon > 0$ there exists $t_0 \in \mathbb{N}$ such that for all $i, j \in N$:

$$|V_i(t) - V_j(t)| < \epsilon$$

for all $t \geq t_0$ and $g(t)$ is connected.

**Example 5.** Consider the same initial network structure and cultural traits with Example 1 and $c^\eta = 0.001$, $c^\Delta = 0.5$. The dynamic of cultural traits is shown in Figure 11.
Figure 10: Dynamic of Cultural Traits for $c^\eta = 0.009, c^\Delta = 0.023$.

Figure 11: Dynamic of Cultural Traits for $c^\eta = 0.001, c^\Delta = 0.5$. 

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Indeed, assume the cost of network changes $c^\Delta$ is sufficiently large, then no agent wants to change links. So the network stays unchanged and cultural traits form a consensus in the limit.

**Example 6.** Consider the same initial network structure and cultural traits with Example 1 and $c^\eta = 0.5$, $c^\Delta = 0.001$. The dynamic of cultural traits is shown in Figure 12.

![Figure 12: Dynamic of Cultural Traits for $c^\eta = 0.5$, $c^\Delta = 0.001$.](image)

Instead, assume the cost of child care $c^\Delta$ is sufficiently large, it means that dynasties will benefit a lot from integration, then all dynasties want to add more links. So it will converge to a complete network and cultural traits also form a consensus quickly.

4.2. The UPST model

Assume that a pairwise stable network with transfers (PST) is reached in each period, i.e., $g(t + 1)$ is a PST network with respect to $g(t)$ and $V(t)$ by definition 2.

4.2.1. General results on convergence

**Theorem 2.** (Convergence) For any given $V(0)$ and $g(0)$, for all $i \in N$, all $\epsilon > 0$ there exists $t_0 \in \mathbb{N}$ such that $|V_i(t) - V_i(t')| < \epsilon$ for all $t, t' \geq t_0$.

---

3In case of the existence of multiple PST networks, a random PST network will be reached.
Proof. Note here the series of the network matrices $g(0), g(1), \ldots$ satisfies the following properties:

(i) $g_{ii}(t) = \frac{n - \eta(t)}{n} > 0, \forall i \in N$;
(ii) $g(t)$ is symmetric for all $t$;
(iii) $\min_{i,j \in N} \{g_{ij}(t)\} \geq \frac{1}{n}$, where $\min_{i,j \in N} \{g_{ij}(t)\}$ stands for the minimum that is taken over all positive entries among $g_{ij}(t)$.

Then by the Stabilisation Theorem proposed in Lorenz (2005), $\exists t_0$ and pairwise disjoint classes of agents $\mathcal{I}_1 \cup \mathcal{I}_2 \cup \ldots \cup \mathcal{I}_p = N$ such that

$$A(\infty) \cdots A(1)A(0) = \begin{bmatrix} K_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & K_p \end{bmatrix}$$

where $K_1, \ldots, K_p$ are quadratic consensus matrices in the sizes of $\mathcal{I}_1, \ldots, \mathcal{I}_p$. Thus

$$V(\infty) = A(\infty) \cdots A(1)A(0)V(0) = \begin{bmatrix} K_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & K_p \end{bmatrix}$$

The convergence of the cultural traits for all agents is guaranteed. \hfill \Box

4.2.2. Simulation results on the dynamics with PST networks

The following example shows that the PST network is not unique in each period.

Example 7. Consider the same initial network structure and cultural traits with Example 2 and $c^n = 0$, $c^\Delta = 0$. The dynamic of cultural traits is shown at Figure 13. As time goes by, dynasty 1 disconnect with all other dynasties. The steady-state cultural traits are $\bar{V}_1 = 1, \bar{V}_i = 5.3, i = 2, \ldots, 10$. $V(0) = (1, 2, 3, 3, 4, 4, 5, 7, 9, 11)$. 

$$g(0) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$
Given $V(0)$ and $g(0)$, both $g^0$ and $g(1)$ are PST networks.

Let us check for $23 \in g(1)$, would 2 and 3 delete the link. The sum of the marginal utilities of deleting the link $23$ of $g(1)$ is

$$
(mu_{2,3})(g(1), 23|V(0), g(0)) = -\frac{(V_2 - V_3)^2}{n^2} - \frac{(2V_3 - V_2 - V_7)^2}{n^2} + \frac{(V_3 - V_7)^2}{n^2}
$$

$$
= \frac{1}{50} > 0.
$$

Therefore 2 and 3 would not delete the link $23$. All the links can be checked likewise to show that $g(1)$ is a PST network.

**Remark 2.** When $c^\eta = c^\Delta = 0$, $g(1)$ only depends on $V(0)$.

**Remark 3.** PST network exists but is not unique at each period.

As either $c^\eta$ or $c^\Delta$ increases enough, the network will converge to a connected (but may be not complete) network, thus all dynasties reach a consensus on cultural traits.

**Example 8.** Consider the same initial network structure and cultural traits with Example 2 and $c^\eta = 0.1$, $c^\Delta = 0.5$. The dynamic of cultural traits is shown at Figure 14. The network converges to a connected network, and all dynasties reach a consensus on cultural traits with the steady-state cultural traits being $\bar{V}_i = 4.9, i = 1, \ldots, 10$.

Let $c^\eta$ be large enough, the network will converge to a connected network, thus all dynasties reach a consensus on cultural traits. Let $c^\Delta$ be large enough, the network will stay unchanged as time goes on.

**Example 9.** Consider the same initial network structure and cultural traits with Example 2 and $c^\eta = 1$, $c^\Delta = 0.5$. The dynamic of cultural traits is shown at Figure 15. The network converges to a complete network, and all dynasties reach a consensus on cultural traits with the steady-state cultural traits being $\bar{V}_i = 4.9, i = 1, \ldots, 10$.

**Example 10.** Consider the same initial network structure and cultural traits with Example 2 and $c^\eta = 0.01$, $c^\Delta = 5$. The dynamic of cultural traits is shown at Figure 16. The network stays unchanged and connected, and all dynasties reach a consensus on cultural traits with the steady-state cultural traits being $\bar{V}_i = 4.9, i = 1, \ldots, 10$.
Figure 13: Dynamic of Cultural Traits for $c^\eta = 0, c^\Delta = 0$.

Figure 14: Dynamic of Cultural Traits for $c^\eta = 0.1, c^\Delta = 0.5$. 
Figure 15: Dynamic of Cultural Traits for $c^\eta = 1$, $c^\Delta = 0.5$.

Figure 16: $c^\eta = 0.01$, $c^\Delta = 5$. 
4.3. The UPS model

Assume the initial state is heterogeneous, i.e., there exist $i$ and $j$ such that $V_i(0) \neq V_j(0)$. If both costs of changing the network and cost of child care are small (relative to the degree of imperfect empathy), then there will be long term heterogeneity.

**Proposition 10.** There exist sufficiently small $c^\Delta$ and $c^\eta$ such that for any $\epsilon > 0$ there exists $t_0 \in \mathbb{N}$ such that for $V(t) := \min_{k \in \mathbb{N}} V_k(t)$ and $\bar{V}(t) := \max_{k \in \mathbb{N}} V_k(t)$, there exists $\bar{V}, \check{V} \in \mathbb{R}$ such that $|V - \bar{V}(t)| < \epsilon$ and $|\bar{V} - \bar{V}(t)| < \epsilon$ for all $t \geq t_0$ and $\bar{V} - \check{V} > 0$.

**Proof.** Denote $x_i$ and $x_j$ as the number of links agent $i$ and $j$ have in $g(0)$, respectively. By Proposition 3, there exist sufficiently small $c^\Delta$ and $c^\eta$ such that the empty network is the unique PS network. Thus $V_i(t) = V_i(0), \forall i \in \mathbb{N}$.

As both $c^\eta$ and $c^\Delta$ increases, more and more dynasties reach a consensus, as shown in Figure 19. Once $c^\eta$ or $c^\Delta$ increases over a threshold, the network will converge to a connected (but may not complete) network, thus all dynasties reach a consensus on cultural traits.

**Example 11.** Consider the same initial network structure with Example 2. The initial cultural traits are $V(0) = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$.

**Small costs**

When $c^\eta = 0.001$, $c^\Delta = 0.001$, the dynamic of cultural traits is shown at Figure 17. The network converges to the empty network, and the steady-state cultural traits are $\bar{V}_i = V_i(0), i = 1, \ldots, 10$. 
Intermediate costs

When \( c^\eta = 0.004, \ c^\Delta = 0.005 \), the dynamic of cultural traits is shown at Figure 18. Dynasties 1, 2, 3, 4, 10 are disconnected with the others, and the steady-state cultural traits are \( \bar{V}_1 = 1, \bar{V}_2 = 2.1, \bar{V}_3 = 3.1, \bar{V}_4 = 4, \bar{V}_{10} = 9.9, \bar{V}_i = 349/50, i = 5, \ldots, 9 \).

When \( c^\eta = 0.005, \ c^\Delta = 0.006 \), the dynamic of cultural traits is shown at Figure 19. Dynasties 1, 4, 10 are disconnected with the others, and the steady-state cultural traits are \( \bar{V}_1 = 1, \bar{V}_4 = 4.2, \bar{V}_{10} = 9.9, \bar{V}_2 = \bar{V}_3 = \bar{V}_i = 5.7, i = 5, \ldots, 9 \).

When \( c^\eta = 0.01, \ c^\Delta = 0.02 \), the dynamic of cultural traits is shown at Figure 20. Dynasty 1 is disconnected with the others, and the steady-state cultural traits are \( \bar{V}_1 = 13/10, \bar{V}_i = 179/30, i = 2, \ldots, 10 \).

Large costs

When \( c^\eta = 0.04, \ c^\Delta = 0.05 \), the dynamic of cultural traits is shown at Figure 21. The network converges to the connected network, and the steady-state cultural traits are \( \bar{V}_i = 11/2, i = 1, \ldots, 10 \).

Large \( c^\Delta \)

When \( c^\eta = 0.07, \ c^\Delta = 4.9 \), the dynamic of cultural traits is shown at Figure 22. The network remains unchanged, and the steady-state cultural traits are \( \bar{V}_i = 11/2, i = 1, \ldots, 10 \).
Figure 18: Dynamic of Cultural Traits for $c^\phi = 0.004$, $c^\Delta = 0.005$.

Figure 19: Dynamic of Cultural Traits for $c^\phi = 0.005$, $c^\Delta = 0.006$. 
Figure 20: Dynamic of Cultural Traits for $c^\eta = 0.01$, $c^\Delta = 0.02$.

Figure 21: Dynamic of Cultural Traits for $c^\eta = 0.04$, $c^\Delta = 0.05$. 
Figure 22: Dynamic of Cultural Traits for $c^\eta = 0.07$, $c^\Delta = 4.9$.

**Large $c^\eta$**

When $c^\eta = 0.1$, $c^\Delta = 0.05$, the dynamic of cultural traits is shown at Figure 23. The network converges to the complete network, and the steady-state cultural traits are $\bar{V}_i = 11/2, i = 1, \ldots, 10$. 
5. Efficiency of networks

All the simulation results of these models showed that for sufficiently small cost parameters, it converges to a heterogeneous society, while for large cost parameters, it converges to a homogeneous society. In this section, we consider the efficiency of networks such that all dynasties would not benefit from deviating to the other networks.

**Definition 5.** A network $g$ is efficient, if $\forall g' \in \mathcal{G}, \sum_{i=1}^{n} U_i(g | V(t), g(t)) \geq \sum_{i=1}^{n} U_i(g' | V(t), g(t))$.

**Definition 6.** A network $g$ is strongly efficient, if $\forall g' \in \mathcal{G}$ and $\forall i \in N$, $U_i(g | V(t), g(t)) \geq U_i(g' | V(t), g(t))$.

The following proposition further shows that the empty network $g^\emptyset$ is the only (strongly) efficient network for small costs and the complete network $g^N$ is the only (strongly) efficient network for large costs.

**Proposition 11.** Assume that the initial cultural traits are heterogenous, i.e., $V_i \neq V_j$, $\forall i \neq j$ and $i, j \in N$. There exist sufficiently small $c^\eta$ and $c^\Delta$ such that the empty network $g^\emptyset$ is the only (strongly) efficient network, and sufficiently large $c^\eta$ such that the complete network $g^N$ is the only (strongly) efficient network.
Proof. Remark that any strongly efficient network is also efficient, thus it suffices to show the argument on strong efficiency. First to show that there exist sufficiently small \( e^\eta \) and \( c^\Delta \) such that the empty network \( g^0 \) is the only strongly efficient network. Fix any \( g(t), V(t) \) and 
\( g \neq g^0 \). It suffices to show that \( \exists e^\eta, c^\Delta \), such that \( U_i(g^0 \mid V(t), g(t)) - U_i(g \mid V(t), g(t)) \geq 0 \), \( \forall i \in N \). Denote \( x_i \) as the number of links that agent \( i \) need to change from \( g(t) \) to \( g^0 \) and 
\( y_i \) the number of links that agent \( i \) need to change from \( g(t) \) to \( g \). Denote the neighborhood of agent \( i \) in \( g \) as \( N_i \) and the degree of \( i \) as \( \eta_i \). Then 
\[
U_i(g^0 \mid V(t), g(t)) - U_i(g \mid V(t), g(t)) \\
= -c^\Delta(x_i)^2 - n c^\eta + \frac{1}{n^2}(d_{iN_i})^2 + c^\Delta(y_i)^2 + c^\eta(n - \eta_i) \\
= c^\Delta(y_i - x_i)^2 - c^\eta \eta_i + \frac{1}{n^2}(d_{iN_i})^2 \geq 0 \\
\iff c^\eta \eta_i + c^\Delta(x_i - y_i)^2 \leq \frac{1}{n^2}(d_{iN_i})^2.
\]
Due to the assumption that \( V_i \neq V_j, \forall i \neq j \) and \( i, j \in N, d_{iN_i} \neq 0 \). Thus such cost parameters always exist to guarantee the empty network \( g^0 \) is the only strongly efficient network.

Then to show that there exist sufficiently large \( e^\eta \) such that the complete network \( g^N \) is the only (strongly) efficient network. Denote \( z_i \) as the number of links that agent \( i \) need to change from \( g(t) \) to \( g^N \). Then 
\[
U_i(g^N \mid V(t), g(t)) - U_i(g \mid V(t), g(t)) \\
= -\frac{1}{n^2}(d_{iN_i})^2 - c^\Delta(z_i)^2 + \frac{1}{n^2}(d_{iN_i})^2 + c^\Delta(y_i)^2 + c^\eta(n - \eta_i) \\
= c^\Delta(y_i - z_i)^2 + \frac{1}{n^2}[(d_{iN_i})^2 - (d_{iN_i})^2] - c^\eta(n - \eta_i) \geq 0 \\
\iff c^\eta \geq \frac{1}{n - \eta_i} \left[c^\Delta(z_i - y_i)^2 + \frac{1}{n^2}[(d_{iN_i})^2 - (d_{iN_i})^2]\right].
\]
Thus such cost parameters always exist to guarantee the complete network \( g^N \) is the only strongly efficient network.

\[\Box\]

6. Conclusion

We studied the dynamics of intergenerational cultural transmission in endogenous networks where the network changes are inherited. We proposed three ways to endogenize the process of network formation. In the first one, the network is supposed to be directed and each dynasty can either form or delete a directed link unilaterally with another dynasty. Therefore, at each period, each family faces a utility optimization problem where a trade-off between own utility losses and the improvements of child’s cultural trait. We have shown that if the cost of network changes is greater than the cost of child care, extremists will never add links, and in the case of sufficiently low cost, extremists may cut all ties with the society. In the second and third models, the network is supposed to be undirected. In the second
model, we assume that after each period, a pairwise stable network with transfers (PST network for short) is reached, i.e., $\forall t \in \mathbb{N}, G(t+1)$ is a PST network for $G(t)$ and $V(t)$. We have shown the existence of the PST network for each period, however, it is not necessary to be unique, evidenced by a counter example. Moreover, a necessary and sufficient condition is given such that a network is PST for given $V(t)$ and $G(t)$. The convergence of cultural traits is guaranteed. In the third model, we assume that after each period, a pairwise stable network (PS network for short) is reached, i.e., $\forall t \in \mathbb{N}, G(t+1)$ is a PS network for $G(t)$ and $V(t)$. In this case, there always exist sufficiently small cost parameters such that the empty network is the unique PS network.

There always exist sufficiently small cost parameters such that the empty network is the unique efficient network, and sufficiently large costs of child care such that the complete network is the unique efficient network.

The dynamics of these three models are studied by both analytics and simulations. For sufficiently small costs of network changes and child care, extremists will disconnect from the other dynasties and there will be a long term heterogeneity of the society. Specially, in the first and the third model, we show that the network will converge to the empty network. While in the second model, the network might or might not converge to the empty network, since the PST network is not unique in each period. As costs of network changes and child care increase, more and more dynasties reach a consensus even though there are still some other dynasties disagree with this consensus. For large costs of network changes and child care, it converges to a homogeneous society such that all dynasties have the same cultural trait in the limit. This give us some insights on how to reduce extremism in our real life. For example, one can consider to foster the interaction of children with different cultural backgrounds such that the cost of network change is increased (extremists will less probably disconnect with others). Some work can also be done to increase value of integration (i.e., increase the benefits from relations). Extremists play an important role in the dynamical process, policy makers should take it into account and provide more opportunities for extremists to connect with others.
References


