Social learning with Bayesian and non-Bayesian agents

Akylai Taalaibekova
Social learning with Bayesian and non-Bayesian agents

Akylai Taalaibekova∗

September 12, 2020

Abstract

We consider a model of social learning, where individuals exchange their opinions about some topic. Agents update their beliefs in a mixed way: in a Bayesian way (as rational players) for private signals and via DeGroot model (as naive players) to take into account the opinions of their neighbors. When all agents form beliefs in this way, there is successful learning in the society. We study how this asymptotic outcome is affected by the presence of Bayesian agents, who take into account only private signals, with the presence of stubborn agents, who do not change initial opinions, and by the presence of polarized agents, who receive biased signals. We show that opinion fluctuations and misinformation appears with the presence of stubborn and polarized agents, respectively. However, the presence of at least one Bayesian agent can bring back the successful learning in the society with stubborn individuals.

We assume that formed beliefs inevitably influence decisions that people take. In this way, stubborn and polarized agents affect optimal actions as well, causing miscoordination in the society. Moreover, we study time to converge to stable opinions and the results state that heterogeneity of agents induces high diversity in timing, highlighting the importance of knowledge of social composition.

Keywords: opinion dynamics, Bayesian learning, DeGroot model, stubborn agents, polarization, social network

JEL Classification: C11, C63, D83

1 Introduction

Everything is constantly updating around the humanity and we need to update along side it. People form and exchange their opinions about milliards of issues. They discuss with

∗CORE, Université catholique de Louvain; Centre d’Economie de la Sorbonne, Université Paris 1 Panthéon-Sorbonne. E-mail: akuka.93@gmail.com
friends, family members, colleagues - people who surround them, namely, with people who constitute their social network. Along with social interaction, people receive, collect, search for information on their own. We can think of external sources of information such as unbiased media sources to which the agent is subscribed (social media, TV, radio, newspapers and etc.). Therefore, there are two channels of information about some given topic: (i) each agent receives information about the subject from an external source and (ii) there is communication between connected agents about the issue. Agents have imperfect information about the topic - the true state of the world, denoted $\theta$.

We consider a model where each individual is embedded in a directed network of relationships. At $t = 0$ agents are endowed with prior beliefs about $\theta$ and at each following period they form and update their beliefs taking into account one or both types of information. Private signals are binary, such that they are either positive “1” or negative “0”, independent over time and across the agents, conditional on the state. In this context, we study opinion dynamics with the presence of heterogeneous agents. There are four types of agents: i) regular agents update their beliefs in a mixed way: as a combination of a Bayesian way (as rational players) for private signals and the DeGroot model (as naive players) to take into account the opinions of neighbors. When the society consists of only regular agents, we call it homogeneous society. ii) Bayesian agents take into account only private signals as a new information and update their beliefs in a Bayesian way. iii) Stubborn agents never change their initial opinions. iv) There are polarized agents in the society who are represented by optimistic and pessimistic individuals. Optimists have higher frequency of positive, while pessimists receive more negative signals. Like stubborn players, polarized agents do not listen to the opinions of other people.

We start our analysis with homogeneous society and study how beliefs are evolving through time with presence of different agents. Do people converge to a stable opinion, if yes, is it unique for all the agents, in other words, is there a consensus among the agents? Therefore, the main objective of this paper is to understand the impact of heterogeneity in types of agents on social learning.

The results show that in case of homogeneous society, there is successful learning such that everyone agrees on consensus that coincides with true state. It is true if we consider a society with regular and Bayesian agents. However, stubborn behavior precludes convergence, causing persistent opinion fluctuations. Interestingly, we show that the negative effect of stubborn individuals can be overcome by the presence of at least one Bayesian agent. In other words, persistent disagreement that stubborn agents cause is precluded by a rational agent. It means, the Bayesian agent effectively aggregates the dispersed information and since she has followers (people who listen to her or trust her opinion) the “correct” information is spread throughout the network. The analysis of the social opinion with polarized agents states that we observe the convergence of beliefs, but there is obviously misinformation in the society. Bias brought by polarized agents, affects asymptotic beliefs of all the agents. The extent of the misleading depends on the level of polarization and trust weights that agents put on polarized agents’ opinions. Interesting to note, if
an individual trusts optimistic and pessimistic agents equally, the negative effect can be discarded.

Apart of opinion dynamics, we consider a decision-making problem, since formed beliefs inevitably influence the actions that agents take. We analyze the effect of information flow on agents’ behavior. In this way, we incorporate the belief of the agent in the utility function. Each agent decides how much effort or time she wants to put, depending on the information she has. We introduce a notion of a reference group that is a group of people the agent wants to coordinate with. We allow the reference group to be different from the set of neighbors, such that the agent communicates with all her neighbors while she might coordinate with only few of them (close friends, family members, etc.). One can think of an example where family members coordinate to spend time together on volunteering, while discussing this topic with all their neighbors or friends. At time $t$, agent $i$ takes an action $x_{i,t}$ that is continuous and between 0 and 1 (for example, a fraction of free time). She wants to minimize the distance between $x_{i,t}$ and average choice of $i$’s reference group, and, at the same time, she wants to be consistent with her own belief $y_{i,t}$. And, finally, there is an individual incentive to take a particular action, that is adjusted by parameter $\gamma$.

Our results show that in the homogeneous society the convergence of actions always occurs, independently of the network structure. Long-run action is determined by individualistic parameter $\gamma$ and true state $\theta$. In this way, as long as $\gamma$ stays identical for all agents, there is full coordination in actions. When there are Bayesian and stubborn agents, despite the successful information aggregation, we observe a miscoordination. Agent’s steady action is determined by the number of stubborn players, distance between the true state and their opinions and their willingness to coordinate with each other. These terms cause a shift from the action in homogeneous society, concluding that there is a possibility to overcome a stubborn behavior in information processing, but it is not true for the action choice. Miscoordination in actions is present for the society with polarized agents as well. Their actions deviate from the one in homogeneous society and are determined by trust weights on optimists and pessimists, willingness to coordinate with them and level of polarization. Consequently, social structure is crucial and has different effect for limit beliefs and long-run actions.

These results can help us to explain election outcomes, volunteering activities, fashion trends and food mainstreams, where information about the subject, decisions of friends and personal preferences or benefits influence the choice that the agent takes. Indeed, in our midst, some people may ignore new information or may receive distorted information. Our results show how such a behavior affect the decisions of the whole society.

We conclude the paper with the analysis of time that the individuals need to reach their stable opinions. We find this question reasonable since one can give a range of examples where communication is time-consuming and, therefore, it can either discard the outcome or make it less valuable. We contribute in the study of speed of convergence of the social learning by providing explicit expressions for time that average opinion reaches steady
state. They are in line with vast literature that stresses the role of the social structure and the second largest eigenvalue of the communication matrix. In addition, we show the impact of initial and asymptotic beliefs. Together with numerical results, we can conclude that diversity in social structure significantly affects the time that agents need in order to converge to their asymptotic beliefs. In particular, stubborn agents have clear negative effects, slowing down the time to converge to the true state.

Our model can be applied to a well-known example concerning the public health. Imagine there is a society of individuals to whom a new vaccination program has been introduced. The willingness to vaccinate depends on individual beliefs of its safety, benefits and consequences. Each individual starts with her own initial opinion on the topic, at each period she discusses and shares her point with neighbors. Moreover, she receives different brochures about the program, she might watch documentary movies about the disease and the immune system, she consults with her personal doctor or read out some encyclopedias. These events shape her opinion at the end of each period. Surely, in real life, people are heterogeneous, we can meet a person who is extremely confident in own opinion. She can ignore opinions of others, the vaccine program and the latest advances in medicine, deeply believing in grandma’s words and her proven recipes. In the literature such a person is called a ”stubborn” one meaning that she is not changing her opinion at all. At the same time, there might be someone who is well-informed, open to new discoveries, follows trusted information channels, but does not trust people’s opinions. In our framework, they represent Bayesian agents. If the processed information is biased towards or against the vaccination, the person is polarized, such that she is optimistic about the vaccination or, in contrary, unreasonably pessimistic. Obviously, the society is better of when people take their optimal actions and make it as soon as possible.

The main focus and a starting point of the work is opinion dynamics, which is a base for the whole paper. Results of the asymptotic beliefs are traced in all subsequent conclusions. First, we provide the analysis of opinion dynamics in different societies. Our contribution is in study of combination of mixed updating rule with several well-known behaviors, where the result with at least one Bayesian agent who is able to bring back successful learning with the presence of stubborn agents can definitely enrich the understanding of social learning processes. Second, we show how asymptotic actions are affected by formed opinions and how they differ with the presence of particular agents. Third, in order to provide a full picture, we give the theoretical results for time to converge supported by numerical exercises.

The rest of the paper is organized as follows. Section 2 comments further on the related literature, Section 3 presents opinion formation model and its results, Section 4 analyses the impact of asymptotic beliefs on decision problem, Section 5 studies how long does it take for the beliefs to converge to a stable one, and Section 6 concludes with a discussion of the results and of possible extensions. The theoretical background and proofs are grouped together in Appendix.
2 Literature review

The topic of modeling and analyzing opinion formation and diffusion received a lot of attention in the literature [see the surveys by Jackson (2008), Acemoglu et al. (2011b), Bramoullé et al. (2016), Golub and Sandler (2016)]. There are two prominent methods of modeling social learning through networks: (1) Bayesian learning, where agents use Bayes’ rule to assess the state of the world and (2) DeGroot learning, where agents instead consider a weighted average of their neighbors’ previous period opinions or actions. The first one considers all agents as fully rational ones and focuses on a game with incomplete information and characterizes its equilibria [see Banerjee et al. (1992), Acemoglu et al. (2011a)]. The second approach was introduced by DeGroot (1974). There are many extensions of the standard model which provide conditions on the network structure under which there is convergence of opinions and characterize steady-state solutions [see Grabisch et al. (2018)].

To the best of our knowledge, there are not many papers that are mixing these two approaches. Jadbabaie et al. (2012) were the first who developed a dynamic model of opinion formation using both Bayes rule and naive learning. Their model combines both personally received information and a social interaction. In such a way, individuals update their beliefs taking their personal signals in a Bayesian way and averaging opinions of their neighbors as in DeGroot model. This approach can be seen as an extension of these learning models making them more flexible and reasonable for interpretation. Fernandes (2018) and Azzimonti et al. (2018) use this update rule to analyze the opinion formation with the presence of confirmatory bias and fake news, respectively. Another mixed approach is studied by Anunrojwong et al. (2018). The authors introduce the signal quality into the naive learning: signal quality is communicated by using Bayes’ rule to combine one’s belief with the beliefs of one’s neighbors. They found that given initial conditions the consensus belief can be reached and provide results for the relationship between informativeness of the signal and agent’s centrality [see also Jiménez-Martínez (2015)].

We base our model on the work by Fernandes (2018). A feature that distinguishes our model from his is the focus on the impact of standard Bayesian, stubborn and polarized agents. Moreover, individuals in our model are facing decision problem and we study the influence of formed opinions on the taken actions. In addition, we study the time needed to reach a stable opinion for agents in different societies. These features of the model help us to understand crucial characteristics of players, influence on each other, their role in opinion formation, decision making process and time to the steady state. We can also draw a parallel with the paper by Azzimonti et al. (2018) that investigates misinformation and polarization in the society with internet-bots (stubborn players) who hold and spread extreme views. We adopt polarized agents and allow them to be of different levels of polarization. Our results hold for any number of polarized agents in the society. Furthermore, we know that their influence determines long run beliefs and
utilities. Unlike the previous papers, we provide the investigation of time to converge to the stable opinions. We conduct this analysis in order to understand what is the expected time that the society needs in order to spread new technology/information and to reach the steady state depending on the social composition. Another work by Acemoglu et al. (2013) also focuses on the impact of stubborn players on the opinion dynamics in the social network. They find that the presence of stubborn agents with opposing opinions precludes convergence to consensus; instead, opinions converge in distribution with disagreement and fluctuations. Our result with stubborn players is in line with their model. The impact of stubborn individuals is also studied by Della Lena (2019). In this model the steady state opinion is a linear combination of the underlying state and stubborn opinion, while our result says that the presence of stubborn players makes the asymptotic beliefs to fluctuate. The reason is that in paper by Della Lena (2019) stubborn agents are assumed to receive signals, though always the same. In contrary, we assume that stubborn agents stay ignorant to any new information. In addition, he provides the analysis of the speed of convergence, but without any stubborn agent. In contrary, the goal of this paper is to study the effect of heterogeneous agents on social processes, and in particular, the timing. As a result, heterogeneity brings significant time increase in information aggregation.

Close to our work is the paper by Mueller-Frank (2014). He considers Bayesian and non-Bayesian communication as well, however, the difference comes in the way we define them. According to Mueller-Frank (2014), Bayesian agents draw fully rational inferences, they have sufficient knowledge about the private information and updating behavior of everyone else, while we define Bayesian agents as the ones who take their private signals in a Bayesian way and it is the only source of information. At the same time, in our framework non-Bayesian agents are as in Jadabaie et al. (2012), while in the paper by Mueller-Frank (2014) they form opinions as in DeGroot (1974). These crucial differences lead us to an interesting conclusion: Mueller-Frank (2014) finds that the presence of at least one Bayesian agent is sufficient for Bayesian and non-Bayesian agents to perfectly aggregate the private information of all agents. We, in turn, say that non-Bayesian agents can successfully aggregate the dispersed information by themselves, hence, the presence of Bayesian agents does not change the limit beliefs. However, we find that at least one Bayesian is enough to make Bayesian and non-Bayesian agents converge to the true state when there are stubborn players in the society as well. Consequently, though defined in different ways, the work of Mueller-Frank (2014) and our results show that even one Bayesian has significant impact on social asymptotic beliefs.

Time to steady state was studied in the paper by Golub and Jackson (2012). They focus on homophily and its relation to speed of convergence. They find that convergence to a consensus is slowed down by the presence of homophily, but is not influenced by network density. A work close to ours is a paper by Jadabaie et al. (2013), where they use the model of Jadabaie et al. (2012) to study the speed of social learning in the presence of information heterogeneity. They analyze the impact of information distribution on the rate at which agents learn the truth. Indeed, their conclusion is - the rate depends on
the topology of the network as well as the agents’ signal structure, such that the rate of
learning increases as agents receive more informative signals. Concerning our paper, we
confirm that heterogeneity in agents’ information affects the timing. Both empirical and
theoretical results show that with presence of heterogeneity, time to convergence does not
depend on the characteristics of the whole society any more, but only of particular agents.
In addition to network topology and signal structure, we stress the role of initial beliefs
of agents. In line with the literature, we state that stubbornness negatively affects time
to convergence, sufficiently increasing the time for the ones who trust them dispro-
portionately. The analysis of convergence speed is also met in work by Förster et al. (2016)
where they show that manipulation slows down the convergence, Olcina et al. (2018) find
that fast convergence can be obtained by large enough conformity compared to spectral
homophily.

A paper by Olcina et al. (2018) also investigates asymptotic behavior. The authors
study the dynamics of assimilation where the social norms of individuals are based on
DeGroot (1974) model. In the long run the society have convergence of norms. In our
model the agents face similar decision problem, but we focus on the impact of heterogene-
ity of agents on the optimal action and utility levels. We found that stubbornness and
polarization significantly affect the actions taken by the agents.

3 Opinion formation

The society consists of a set \(N = \{1, \ldots, n\}\) of agents who at discrete time \(t = \{1, 2, \ldots\}\)
obtain some information and discuss it with each other. Let \(\Theta = [0, 1]\) is the set of
possible states of the world. States can be considered as possible values or estimates of
some issues. One can find many real-world applications in various fields. For example, in
engineering they need to estimate the proportion of structural defects after fabrication,
in social science - to estimate the proportion of individuals who would respond “yes” on
a census question, in medical science - to estimate the proportion of patients who make a
full recovery after taking an experimental drug to cure a disease and others. One can also
think of \(\theta \in \Theta\) as a benefit level of vaccination, where \(\theta = 0\) says it gives no benefit or has
the lowest value of safety, while \(\theta\) close to 1 indicates that vaccination is necessary and
protects the immune system. Note, the realized state or true value of \(\theta\) is not observed
directly by the agents.

Belief distribution In order to model the beliefs, we will use a probability distribu-
tion called the Beta distribution with shape parameters \(\alpha, \beta > 0\). A player \(i\) at time \(t\) has
an opinion \(y_{i,t} \in [0, 1]\) that is formed by parameters \(\alpha_{i,t}\) and \(\beta_{i,t}\) such that:

\[
y_{i,t} = \mathbb{E}[\theta] = \frac{\alpha_{i,t}}{\alpha_{i,t} + \beta_{i,t}}^1
\]

\(^1\)The opinion value is taken as a mean of the Beta distribution. It is a real number that summerizes
“well” the whole belief (see Appendix 7.1).
Consequently, each agent $i$ begins with a prior belief $y_{i,0}$ that describes her (subjective) belief about the parameter $\theta$ at period $t = 0$. We call $y_i$ a column vector with $n$ opinions, where $y_i^T$ is a transpose of opinion vector. Depending on the initial parameters of the belief, $\alpha_0$ and $\beta_0$, agents can start from different points of views.

The figure demonstrates how beta distribution is formed depending on the parameters. Vertical red lines represent a mean value of the distributions, hence, the opinion of the agent. One can conclude that depending on the values of $\alpha$ and $\beta$, the agent holds low or high estimate about $\theta$. Further, we will see the impact of initial distributions on the formation of long-run opinion with the presence of heterogeneous agents.

**Social interaction** Agents communicate with each other and update their opinions. We say that individuals are embedded in a fixed directed network. Interaction among the agents constitutes an adjacency matrix $W^2 = [W_{ik}]$ with $i, k \in N$ that represents the relative importance that each individual assigns to the opinions of all members in the society, including herself. More precisely, $W_{ik}$ denotes the weight or trust that individual $i$ assigns to the current opinion of agent $k$ in forming her own opinion in the next period. We do not impose any symmetry on the links in the network so that we allow $W_{ik} \neq W_{ki}$. This simply means that any two agents can give different weights to each other. By $N_i = \{j \in N : W_{ij} > 0\}$ we denote the set of neighbors of player $i$ in the network and we denote by $d_i$ the number of outgoing links of $i$, i.e., $d_i = |N_i|$. It is assumed that $W$ is a row stochastic matrix, i.e., $\sum_{k=1}^{n} W_{ik} = 1$ for every $i \in N$.

The network structure is represented by a directed graph denoted by $G$. We define a path from $k$ to $i$ in $G$ as a sequence of agents starting with $k$ and ending with $i$ such that each agent is a neighbor of the next agent in the sequence. The social network is strongly connected if for every pair of individuals $i, k \in N$ there exists a path from $i$ to $k$.

---

\[^2\]We use bold uppercase letters to denote matrices, while its components can be accessed by subscripts, for example, $W_{ij}$ of matrix $W$.

\[^3\]|A| denotes the cardinality of a set $A$. 

8
**External signals** We assume that agents not only listen to the opinions of their neighbors, but they can also receive external information. They can observe another sources to obtain additional knowledge. This type of source of agent $i$ is presented by the private signal $s_{i,t} \in S = \{1, 0\}$ at each period $t$. Signals are independent over time and across the agents, conditional on the state. When the true parameter value is $\theta$, agent $i$’s information source delivers signal $s_{i,t}$ with probability $\theta$. A signal at each period follows Bernoulli distribution:

$$s_{i,t} \sim \text{Bernoulli}(\theta)$$

At every period a new signal is realized, agent $i$ gets signal $s_{i,t} = 1$ with probability $\theta$ and signal $s_{i,t} = 0$ with probability $(1 - \theta)$. We present the signal structure for agent $i$ at time $t$ in the following form:

$$s^1_{i,t} = 1 \{s_{i,t} = 1\}$$
$$s^0_{i,t} = 1 \{s_{i,t} = 0\}$$

(3.2)

The following signal structure is inspired by the model of Fernandes (2018). The author studies binary information with the consequences of ambiguous signals. In our model we omit the ambiguity part and focus only on binary signals.

**Updating beliefs** Our updating rule is based on Jadbabaie et al. (2012). They propose the rule where an individual updates her belief as a convex combination of Bayesian posterior belief conditioned on her signal and the opinions of neighbors. Later, this approach can be seen in papers by Fernandes (2018) and Azzimonti et al. (2018). In this way, following the realization of the signals, each agent computes her Bayesian posterior belief conditioned on the received private signal, and then sets her final parameters $\alpha$ and $\beta$ as a linear combination between Bayesian posteriors and the weighted average of neighbors. After receiving signals, the players form their beliefs (see Appendix 7.1):

$$\alpha_{i,t+1} = W_{ii}[\alpha_{i,t} + s^1_{i,t+1}] + \sum_{k \neq i}^n W_{ik}\alpha_{k,t}$$
$$\beta_{i,t+1} = W_{ii}[\beta_{i,t} + s^0_{i,t+1}] + \sum_{k \neq i}^n W_{ik}\beta_{k,t}$$

(3.3)

where $[\alpha_{i,t} + s^1_{i,t+1}]$ and $[\beta_{i,t} + s^0_{i,t+1}]$ are Bayesian posteriors for parameters $\alpha_{i,t+1}$ and $\beta_{i,t+1}$, respectively. We explore the relation between the evolution of opinions about some common parameter of interest $\theta$, different characteristics of players and network structure. Note, if agent $i$ trusts only herself such that $W_{ii} = 1$ then she discards the social influence and behaves as Bayesian agent. This type of agents will be addressed in the next subsection where we study the role of rational agents.

In order to see if agents successfully process coming information, we introduce the notion of a **consensus**.
Definition 1. We say that a society reaches a consensus given initial beliefs $y_0$ if there exists $y$ such that for a small $\epsilon > 0$

$$\lim_{t \to \infty} |y_{i,t} - y| < \epsilon \text{ for all } i \in N$$

It means that in the long run the opinions of all agents converge to $y$ and once it is reached, the opinions remain the same.

3.1 Homogeneous society

We start with a society of $n$ agents who are interacting with each other and receiving private signals. At each period they update their opinions as in (3.3) by taking into account both social interaction and constantly coming new external information. We refer to these agents as regular and define $N_r$ as a set of regular agents, such that $N_r \neq \emptyset$. A society is homogeneous when $N = N_r$, since all agents update their beliefs in the same way.

Proposition 1. (Special case of Proposition 5 in Fernandes (2018)) If $G$ is strongly connected, $N = N_r$, then for $i \in N$ opinion $y_{i,t}$ in the long run converges to $\theta$, such that

$$\lim_{t \to \infty} y_{i,t} = y_{i,\infty} = \theta.$$  

When all agents use an updating rule (3.3), they will eventually learn the true state in the long run. It is in line with works by Jadbaiae et al. (2012), Azzimonti et al. (2018) and Fernandes (2018), where in a strongly connected network as long as the individuals take the signals into Bayesian way, repeated interactions lead them to successfully aggregate information and learn the true parameter $\theta$. The convergence occurs independently of the network structure. Moreover, in contrast with DeGroot (1974) and fully Bayesian learning, in our model with both Bayesian and DeGroot opinion updating, asymptotic beliefs are independent of their vector of initial opinions (priors).

Example 1. Consider a society of 10 agents who are interacting with each other in a fixed directed network (Figure 1). In order to avoid the complexity in graph representation, we skip self loops. However, they are present for all the agents, since $W_{ii} \neq 0$, for all $i \in N$, in other words, they put positive trust weights on themselves (in order to account for the external signals). They place different trust weights to each of their neighbor’s opinion (Figure 2). Note, the social network is strongly connected that is crucial for Proposition 1. We would like to add that the arrows on Figure 1 indicate the direction of information flow or who trusts whom. For example, there is an arrow from the agent 10 to the agent 1 that means that the agent 10 listens to or trusts the agent 1.
Each agent starts with initial parameters $\alpha_{i,0}$ and $\beta_{i,0}$ that form her prior belief $y_{i,0}$ about the topic according to (3.1):

<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{i,0}$</td>
<td>0.75</td>
<td>2.14</td>
<td>0.46</td>
<td>0.9</td>
<td>1.06</td>
<td>0.60</td>
<td>0.83</td>
<td>0.37</td>
<td>0.90</td>
<td>0.76</td>
</tr>
<tr>
<td>$\beta_{i,0}$</td>
<td>0.47</td>
<td>0.19</td>
<td>0.92</td>
<td>0.12</td>
<td>0.39</td>
<td>0.77</td>
<td>0.12</td>
<td>2.60</td>
<td>1.88</td>
<td>0.26</td>
</tr>
<tr>
<td>$y_{i,0}$</td>
<td>0.61</td>
<td>0.91</td>
<td>0.33</td>
<td>0.88</td>
<td>0.73</td>
<td>0.44</td>
<td>0.87</td>
<td>0.12</td>
<td>0.32</td>
<td>0.74</td>
</tr>
</tbody>
</table>

Initial opinion of agents

According to the table, the agent 2 starts with the most optimistic view, the agent 4 holds a high estimation as well, while the agents 3, 8 and 9 are more pessimistic about the topic with initial beliefs 0.33, 0.12 and 0.32, respectively.

Individuals receive private signals, that are either positive (“1”) or negative (“0”), given particular state value $\theta$. For the current example, let us assume that $\theta$ equals to 0.5.

The figure illustrates opinion dynamics of 10 people with $\theta = 0.5$. We see that the society successfully aggregates the dispersed information and convergence emerges independently of initial beliefs.
Let us think of a climate change topic. There is a group of agents where each one starts with different opinion. They discuss together the topic and at the same time observe private information (media sources, scientific articles, etc.): positive or negative. As Figure 3 shows, the opinions coincide and converge to the true state $\theta = 0.5$. In the long run, each agent holds true information about the climate change topic.

Successful learning emerges when all agents update their beliefs in the same way. However, it is might be far from real-life situations where the heterogeneity dominates. We would like to capture well-known behaviors in the literature such as stubbornness and polarization in opinions. We know how social opinion evolves when all agents are regular, when the society is homogeneous, so it gives us an obvious reference point to which the society with stubborn, Bayesian and polarized agents can be compared.

### 3.2 Stubborn and Bayesian agents

In this section we consider a society where apart of regular individuals there are individuals who update their beliefs differently, namely, Bayesian and stubborn agents. The goal is to analyze the impact of the heterogeneity in opinion formation on asymptotic social belief.

Let us start the analysis with **Bayesian agents**. In our framework a Bayesian agent updates her belief based on her previous one and new arriving signals. We assume that there are no friends found in her neighborhood, therefore the agent attaches weight 1 to her own information. Unlike standard Bayesians in the literature, in our framework they do not account for information coming from social ties. It is a substantial information circulated in world of mouth that is ignored, but this abuse of definition will allow us to conduct the analysis and obtain interesting results. Note, we consider Bayesian agents, who cannot be considered to be fully rational ones.

Bayesian agents trust only themselves ($W_{ii} = 1$) and take into consideration new coming signals. Therefore, their updating rule (3.3) shrinks to

\[
\alpha_{i,t+1} = \alpha_{i,t} + s_{i,t+1}^1 \\
\beta_{i,t+1} = \beta_{i,t} + s_{i,t+1}^0
\]

which is consistent with Bayesian updating rule based on private signals. We call $B$ a set of Bayesian agents. Need to note that interaction matrix $W$ is not strongly connected any more since there is no path leaving Bayesian agents. According to the theory of absorbing Markov chains, regular agents converge to opinions of Bayesian individuals. We can conclude:

**Proposition 2.** If $N = N_r \cup B$, then for $i \in N$ the opinion $y_{i,t}$ in the long run converges to $\theta$, such that

\[
\lim_{t \to \infty} y_{i,t} = y_{i,\infty} = \theta.
\]

Recall that $N_r \neq \emptyset$ and if $|B| \to 0$, then we have the homogeneous society. As we can see, adding Bayesian agents in the society does not change the long run outcome.
While regular agents are able to process the dispersed information efficiently, the presence of Bayesian agents who spread information based only on informative signals, does not change the asymptotic opinions that are coinciding with the true state.

One can think of Bayesian agent as a community leader. She forms her belief based on the information collected from the experts and credited organizations. However, she does not take into account social discussions, but only announces her opinion to people who listen to her. We believe that such a framework makes sense in real life, especially when the topic is of high risk and uncertainty, and the leader has to focus only on the trusted sources, suspending her social communication channels.

In contrast to regular and Bayesian agents, **stubborn agents** are the ones who do not change their initial opinions. Nevertheless, they still have influence on the social belief. Let us call $S$ a set of stubborn individuals. Each player $j \in S$ does not care about the opinions of other agents', but only follows her own point of view, i.e., puts trust weight only on herself, s.t. $W_{jj} = 1$. Moreover, the stubborn player in our framework does not receive signals. Such kind of person can be very confident in her opinion and ignores any new coming information. The signals are: $s_{jt, t}^1 = s_{jt, t}^0 = 0$ for every $j \in S$ and every $t$.

Updating rule (3.3) for stubborn agents becomes:

$$\alpha_{jt, t+1} = \alpha_{jt, t} = \alpha_{jt, 0}$$
$$\beta_{jt, t+1} = \beta_{jt, t} = \beta_{jt, 0}$$

Imagine a society with regular and stubborn agents. Like in the previous case, interaction matrix $W$ is not connected, with absorbing states presented by stubborn agents. Because of the fact that stubborn agents do not have private signals, the opinion dynamics of regular ones do not converge in the long run.

**Proposition 3.** If $N = N_r \cup S$, then the opinion $y_{i, t}$ of player $i \in N_r$ does not converge to $\theta \in (0, 1)$ in the long run and takes the following expression:

$$\lim_{t \to \infty} y_{i, t} = \lim_{t \to \infty} \frac{\sum_{j \in S} \Pi_{ij} \alpha_{j, 0} + \sum_{l=0}^{CT-1} \sum_{k \in N \setminus S} W_{kk} W_{ik}^{l} s_{k, t-l}^1}{\sum_{j \in S} \Pi_{ij} (\alpha_{j, 0} + \beta_{j, 0}) + \sum_{l=0}^{CT-1} \sum_{k \in N \setminus S} W_{kk} W_{ik}^{l}} = i \in N_r$$

$$\lim_{t \to \infty} y_{j, t} = y_{j, 0} \quad j \in S$$

The expression above contains $\sum_{l=0}^{CT-1} \sum_{k \in N \setminus S} W_{kk} W_{ik}^{l} s_{k, t-l}^1$ that is stochastic and does not vanish. This implies that uncertainty will remain, and priors of stubborn agents will be important in shaping asymptotic beliefs of the whole society. It means, with a presence of agents who do not change their initial opinions, there is no convergence in the society, which means there is no successful learning. The aggregate opinion constantly fluctuates...
depending on the characteristics of stubborn individuals, network structure and signal flows. The result is in line with vast literature that says the presence of stubborn agents precludes the convergence.

**Example 2.** We introduce one stubborn player in the society given in Example 1. We have \( N = N_r \cup S \), where \( |S| = 1 \).

We assume that agent 10 is stubborn, she does not trust anyone except herself. Note, there are no outgoing links, which means \( W \) is not strongly connected. Agent 10 is denoted by red color in the network, while orange color means that agent is a regular one. Graph of opinion dynamics shows constant fluctuations during the whole period for all regular agents, while agent 10 holds her initial belief.

We realize that the current result is discouraging, leaving us with the idea that the successful learning is incompatible with stubbornness. It is true, unless along with stubborn agents there are no aforementioned Bayesian ones.

**Proposition 4.** If \( N = N_r \cup B \cup S \) and \( |B| \geq 1 \), then the opinion \( y_{i,t} \) in the long run evolves in the following way:

\[
\lim_{t \to \infty} y_{i,t} = y_{i,\infty} = \theta \quad i \in N \setminus S
\]

\[
\lim_{t \to \infty} y_{j,t} = y_{j,0} \quad j \in S
\]

Bayesian agents have positive social impact on the aggregate opinion. Moreover, the result states that at least one Bayesian agent is enough to make the opinions of all regular agents to converge to the true state \( \theta \). We conclude that Bayesian agent can effectively aggregate the dispersed information and since she has followers (people who trust her) the “correct” information will be spread throughout the network. Technically it means that agents in the long run converge to the opinions of (absorbing states) stubborn and Bayesian agents. Due to the fact that stubborn agents discard the signals, regular agents
converge only to the asymptotic opinion of Bayesians. Note that $B \neq \emptyset$, otherwise we are back in Proposition 3 where there is no convergence.

Another fact that we would like to highlight is that successful learning emerges independently from the network structure. It means that Bayesian agent does not need to be a central node or to have a substantial influence, even at the edge of the society, with a small audience, she provides convergence to the true state.

Proposition 4 highlights the role of Bayesian agents in shaping the beliefs of individuals in heterogeneous society. Paper by Mueller-Frank (2014) draws a similar conclusion, where the author states that regardless of the number and their position, at least one Bayesian is sufficient to perfectly aggregate the private information of all agents in a mixed society.

Stubbornness is a regular behavior in the society. Refusing to update their beliefs, people might unconsciously prevent the whole society to reveal the true information. Clearly, depending on topic at hand, social consequences might be large and even irreversible. Hence, Proposition 4 states that adding or turning at least one agent into Bayesian, can serve as a tool to provide the successful learning in the society. From now on, we focus on the society with both Bayesian and stubborn agents ($N = N_r \cup B \cup S$).

**Example 3.** The following graph gives the example of opinion dynamics for the society with 3 stubborn agents (red ones) and 1 Bayesian agent (green one), $N = N_r \cup B \cup S$. One can observe that all non stubborn players’ beliefs converge to the true state $\theta$. Note, stubborn and Bayesian agents do not have outgoing links.

![Social network with stubborn agents 7, 8, 9 and Bayesian agent 10](image)

![Opinion dynamics with $\theta = 0.5$](image)

Indeed, opinion dynamics graph shows that agents 7, 8 and 9 are holding their initial beliefs during the whole time period, while beliefs of other agents converge to $\theta = 0.5$.

Compared to Example 2, we add two more stubborn players and there is one Bayesian agent. Despite the increased number of stubborn individuals, there is successful learning in the society, thanks to agent 10. The numerical simulations support the theoretical result.
3.3 Polarized agents

Opinion polarization can be caused by informational frictions. In our model we consider polarized agents who are subscribed to biased sources of information. In another words, we introduce two opposite groups of agents: optimistic and pessimistic agents. We define $P^+$ set of optimists and $P^-$ set of pessimists, such that $P = P^- \cup P^+$ is a set of polarized agents.

Polarized agents behave as Bayesian ones, such that they count only on their own opinions and receive external signals. The crucial difference is in their biased signals. Optimists receive positive signals with higher frequency than the others. In analytical form: $E[s_{j,t}^1] = \theta + \kappa$ for every $t$ and $j \in P^+$. The opposite case is for pessimistic agents, who are prone to receive more negative views about the topic: $E[s_{j,t}^1] = \theta - \kappa$ for every $t$ and $j \in P^-$. Hence, the parameter $\kappa$ indicates the level of polarization in the society.

Proposition 5. If $N = N_r \cup P^+ \cup P^-$, the opinion converges to

$$\lim_{t \to \infty} y_{i,t} = y_{i,\infty} = \theta - \kappa \left( \sum_{k \in P^-} \Pi_{ik} - \sum_{k \in P^+} \Pi_{ik} \right) \quad i \in N_r$$

$$\lim_{t \to \infty} y_{j,t} = \theta + \kappa \quad j \in P^+ \quad \lim_{t \to \infty} y_{j,t} = \theta - \kappa \quad j \in P^-$$

Proposition 5 states there is opinion convergence. Unlike in Proposition 4, the structure of trust relationships is the main determinant for asymptotic beliefs. In this way, each agent ends up with individual belief in the long run. Recall that $\Pi$ is a matrix of long run trust weights and it plays a major role in determining the long run beliefs in the society (see Appendix 7.1).

Example 4. In current example we consider a society with polarized agents $N = N_r \cup P^+ \cup P^-$. We set $\kappa = 0.2$, while $\theta = 0.5$.

![Social network with pessimistic agents 6, 7, 8 and optimistic agents 9, 10.](image)

Figure 4: Opinion dynamics with $\theta = 0.5$, $\kappa = 0.2$. 
This example shows the divergence of asymptotic opinions with 3 pessimistic agents (grey nodes) and 2 optimistic ones (yellow nodes). According to Figure 4, agent 2 holds the highest asymptotic belief among the regular agents. Indeed, looking at the interaction matrix in Figure 2, one can conclude that agent 2 is the one who trusts optimists more than anyone else. The other agents are influenced mostly by the pessimists, it explains the number of opinions that converges close to pessimists’ ones. Hence, the numerical results are consistent with Proposition 5.

4 Decision problem

This section is devoted to decision making problems in the society. We would like to study how asymptotic opinions shape behavior of agents in different societies. For this, we incorporate belief parameters in the decision process and study long-run action dynamics.

Agents’ action Individuals in our model form opinion at each period of time. Moreover, they face a decision problem in which agents’ objective is to maximize the expected utility. At time \( t \), each player \( i \in N \) takes an action, simultaneously with all other individuals, action \( x_{i,t} \in [0,1] \). Action values are in the same range as beliefs, constrained by 0 and 1. It can be interpreted as fraction of time the agent decides to devote to particular activity (sport, study, volunteering, etc.).

Stages of the game:

1. At time \( t = 0 \) each agent \( i \in N \) is endowed with prior belief \( y_{i,0} \) about the subject at hand.

2. At the beginning of each period \( t > 0 \), each agent \( i \in N \) receives a signal \( s_{i,t} \).

3. Agents update their beliefs taking into account received signals and beliefs of their neighbors in the previous period.

4. Agents choose their action \( x_{i,t} \).

5. The process starts again from stage 2.

In our model we capture the idea that taking a decision an individual wants to coordinate with her family members/ friends/ spouse, so it might be that she wants her action to be similar not to all agents in her network (neighbors \( d_i \)), but to particular people. We call these people reference group and define as a set \( R_i \) with \( r_i = |R_i| \)- number of people agent \( i \) wants to coordinate with. Hence, we allow \( r_i \leq d_i \), while \( r_i = d_i \) brings us to a case where the player wants to coordinate will all her neighbors. In order to model this idea, we introduce a new matrix \( R \). For reference group we do not allow for self-loops so that \( r_{ii} = 0 \). We set the weights \( r_{ij} = \frac{1}{r_i} \) if \( j \in R_i \) and \( r_{ij} = 0 \) otherwise.

\[
    u_{i,t}(x_{i,t}) = \gamma x_{i,t} - \frac{1}{2}(x_{i,t} - y_{i,t})^2 - \frac{1}{2}\mu(x_{i,t} - \sum_{j \in R_i} r_{ij}x_{j,t})^2
\]

(4.1)
The utility function consists of three parts that are responsible for different motivations of the agents when they have to make decisions. First of all, agents always have their own incentives to take this or any other option. It can be seen as financial outcomes or personal benefits. Therefore, expression $\gamma x_{i,t}$ stays for capturing utility from different action choices where larger $\gamma$ gives higher utility. Second expression $\frac{1}{2}(x_{i,t} - y_{i,t})^2$ implies that the agent wants to stay consistent with the belief she holds. And the last term, as we already discussed, is for a coordination with people the agent is close to. Parameter $\mu$ stands for the coordination in the society. The higher $\mu$ is, the stronger is the willingness of people to coordinate with each other. The strategy of each agent entails the optimal action $x_{i,t}$ in order to maximize the expected utility function. Thus, solving for the first order condition:

$$x_{i,t}^* = \frac{1}{1 + \mu} \gamma + \frac{1}{1 + \mu} y_{i,t} + \frac{\mu}{1 + \mu} \sum_j r_{ij} x_{j,t}^*$$

Presenting this expression in matrix form:

$$x_t^* = \left[ I - CR \right]^{-1} B(\gamma + y_t)$$

(4.2)

where $y_t$ is a vector with beliefs $y_{i,t}$ for all $i \in N$, $B$ and $C$ are matrices with diagonal elements $\frac{1}{1 + \mu}$ and $\frac{\mu}{1 + \mu}$, respectively. The goal is to explore the relationship between the evolution of opinions about some common parameter of interest and the actions agents are facing. Besides of that, there are new parameters such as coordination parameter $\mu$, reference group $R$ and individualistic value $\gamma$ that are influencing the decision making process.

Similar framework is studied in the paper by Olcina et al. (2018), where they consider an assimilation problem in the society. In our framework we allow for a distinct and flexible reference group, it is done in order to distinguish the effect of information flow and coordination incentives on the action choice. In addition, we analyze the heterogeneity in the composition of the society.

In order to simplify the expression of $x_t^*$, we call $D = \left[ I - CR \right]^{-1}$. Need to note, matrix $R$ has its largest eigenvalue equal to 1 and all entries of matrix $C$ are smaller than 1, therefore $D$ is invertible. The value $D_{ij}$ is higher if agent $i$ is coordinating with agent $j$. However, even though $j \notin R_i$, $D_{ij}$ still can be positive due to the network externalities, that is: if agent $i$ is not coordinating with agent $j$, $i$ is affected by agent $j$ through her coordinating peers.

**Proposition 6.** In the long run the optimal action for agent $i$ in:

1) $N = N_r$: $x_{i,\infty}^* = \gamma + \theta$

2) $N = N_r \cup B \cup S$: $x_{i,\infty}^* = \gamma + \theta - \frac{1}{1 + \mu} \sum_{j \in S} D_{ij}(\theta - y_{j,0})$
iii) \( N = N_r \cup P^+ \cup P^- \):

\[
x^*_i = \gamma + \theta + \frac{\kappa}{1 + \mu} \sum_{j \in N} D_{ij} \left( \sum_{k \in P^+} \Pi_{jk} - \sum_{k \in P^-} \Pi_{jk} \right)
\]

Proposition 6 provides us the optimal long-run actions of the agents in different societies. Result i) describes a homogeneous society and says there is a full coordination in actions. It is true as long as the value of \( \gamma \) is set to be identical for all individuals. Moreover, neither the structure of interaction network nor the reference group play a role here. Note also, the actions are increasing with higher frequency of positive signals (higher \( \theta \)). For example, if human contribution in global warming is high and the agent \( i \) reveals it correctly, she puts corresponding high effort to reduce her own effect.

Result ii) is for a society where along with the regular agents there are also stubborn and Bayesian ones. Though they have different asymptotic beliefs, the expression for the optimal long-run action is true for all \( i \in N \). The difference comes in value \( D_{ij}(\theta - y_{j,0}) \) that is higher for stubborn individuals than for the others. As a result, they have larger deviation from action in i). The action depends on the number of stubborn individuals and the distance \( \theta - y_{j,0} \). Note, it can take positive and negative values. The deviation from the optimal action can be in both directions, up and down, scaled by \( D_{ij} \). Obviously, agent \( i \) is further from \( (\gamma + \theta) \) if she is coordinating with stubborn agents.

For result iii), for a society with polarized agents, there is an additional parameter \( \kappa \) that regulates how far is the action from the one in i). Clearly, with \( \kappa \to 0 \), agents are taking actions as in i). The action choice depends also on trust weights: how much agent \( i \) trusts optimistic and pessimistic agents’ opinion. According to Proposition, if she trusts them equally, their effect vanishes.

Example 5. We would like to illustrate the dynamics of optimal actions for three studied societies: Example 1 - a homogeneous society, Example 3 - a society with 3 stubborn and 1 Bayesian agents and as in Example 4 - a society with 3 pessimistic and 2 optimistic agents. For the numerical exercise we use: \( \gamma = 0.3, \mu = 1, \kappa = 0.2, \theta = 0.5 \), as well as

\[
R = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0
\end{bmatrix}
\]

We give an example of a reference group matrix \( R \). Note, for \( x^*_{i,t} \) we normalize the weights.
We can see from the graphs that, indeed, in the left graph there is a full coordination, while the central and the right graphs illustrate a divergence of actions. According to $R$, there are people who coordinate with stubborn agents, that is why, in the central graph, actions of non-stubborn agents are affected by a stubborn behavior in the society. Note, that agents 7, 8 and 9 who are stubborn, have larger deviation compared to other players. The society with polarized agents, as in the case of opinion dynamics, has the asymptotic actions between the optimists’ (top) and pessimists’ (bottom) ones, the actions of the regular agents are spread in-between.

Accordingly, the optimal utility functions differ for these societies.

**Proposition 7.** In the long run an optimal utility of agent $i$ in:

1) $N = N_r$:

$$u_i(x^*_\infty) = \frac{\gamma^2 + 2\gamma \theta}{2}$$

2) $N = N_r \cup B \cup S$:

For agent $i \in N \setminus S$

$$u_i(x^*_\infty) = \gamma^2 + 2\gamma \theta - \frac{1}{2(1 + \mu)} \left( \sum_{j \in S} D_{ij}(\theta - y_{j,0}) \right)^2 - \frac{\mu}{2(1 + \mu)^2} \left( \frac{1}{r_i} \sum_{k \in R_i} \sum_{j \in S} D_{kj}(\theta - y_{j,0}) \right)^2 +$$

$$+ \frac{\mu}{(1 + \mu)^2} \left( \frac{1}{r_i} \sum_{k \in R_i} \sum_{j \in S} D_{kj}(\theta - y_{j,0}) \right) \left( \sum_{j \in S} D_{ij}(\theta - y_{j,0}) \right)$$

For agent $i \in S$

$$u_i(x^*_\infty) = \gamma^2 + 2\gamma y_{i,0} - \frac{1}{2(1 + \mu)} \left( \sum_{j \in S} D_{ij}(\theta - y_{j,0}) \right)^2 - \frac{\mu}{2(1 + \mu)^2} \left( \frac{1}{r_i} \sum_{k \in R_i} \sum_{j \in S} D_{kj}(\theta - y_{j,0}) \right)^2 +$$

$$+ \frac{\mu}{(1 + \mu)^2} \left( \frac{1}{r_i} \sum_{k \in R_i} \sum_{j \in S} D_{kj}(\theta - y_{j,0}) \right) \left( \sum_{j \in S} D_{ij}(\theta - y_{j,0}) \right) -$$

$$- \frac{(y_{i,0} - \theta)^2}{2} - \frac{y_{i,0} - \theta}{1 + \mu} \left( \sum_{j \in S} D_{ij}(\theta - y_{j,0}) \right) - \frac{(y_{i,0} - \theta)^2}{2} - \frac{y_{i,0} - \theta}{1 + \mu} \left( \sum_{j \in S} D_{ij}(\theta - y_{j,0}) \right)$$
iii) $N = N_r \cup P^+ \cup P^-$:

$$u_i(x^*_i) = \frac{\gamma^2 + 2\gamma y_{i,\infty}}{2} - \frac{\kappa^2}{2(1+\mu)} \left( \sum_{j \in N} D_{ij} \left( \sum_{k \in P^+} \Pi_{jk} - \sum_{k \in P^-} \Pi_{jk} \right) \right)^2 - \frac{\mu \kappa^2}{2(1+\mu)} \left( \sum_{j \in N} \sum_{l \in N} D_{ij} \left( \sum_{k \in P^+} \Pi_{lk} - \sum_{k \in P^-} \Pi_{lk} \right) \right)^2 +$$

$$+ \frac{\mu \kappa^2}{(1+\mu)^2} \left( \sum_{j \in N} D_{ij} \left( \sum_{k \in P^+} \Pi_{jk} - \sum_{k \in P^-} \Pi_{jk} \right) \right) \left( \sum_{j \in N} \sum_{l \in N} D_{ij} \left( \sum_{k \in P^+} \Pi_{lk} - \sum_{k \in P^-} \Pi_{lk} \right) \right) -$$

$$- \frac{(y_{i,\infty} - \theta)^2}{2} + \frac{\kappa(y_{i,\infty} - \theta)}{1+\mu} \left( \sum_{j \in N} D_{ij} \left( \sum_{k \in P^+} \Pi_{jk} - \sum_{k \in P^-} \Pi_{jk} \right) \right)$$

In result i), a full coordination in actions provides us a full coordination in utilities. Like the optimal action, the utility function depends only on $\gamma$ and $\theta$, increases with higher $\theta$ and ignores the structure of the network.

In ii) we distinguish two results: for regular and Bayesian agents ($i \in N \setminus S$) and stubborn ones ($i \in S$). Such a difference in results comes from the fact that agent $i \in N \setminus S$ bases her decision on her asymptotic belief $y_{i,\infty} = \theta$, while agent $j \in S$ receives the utility according to $y_{j,\infty} = y_{j,0}$.

The first part in the utility function coincides with the result i). Agents would get it if they are all regular, but in the current situation, they have to subtract the influence from stubborn agents, first, on themselves and, second, on their coordination peers. Moreover, as the expression shows, agents have to add/subtract (depending on $\theta - y_{j,0}$) a combination of the two influences. This is true for all non-stubborn agents, while stubborn individuals, apart of the influence described above, have a term that depends on the distance between her initial opinion and the true state. The smaller this distance is, the closer the action to the optimal one the agent takes. Evidently, if $y_{i,0} = \theta$, then two expressions coincide. Note, that if $\mu \to 0$, such that people want to coordinate less with each other, than influence of neighbors and a mixed one are vanishing, while agent $i$ still accounts for an impact of stubborn individuals on herself.

First of all, let us look at several novelties coming with result iii). One can easily notice that the utility depends on the matrix $\Pi$, where $\Pi_{jk}$ simply means how much influence agent $k$ has on the long-run behavior of agent $j$. Hence, agents take into account the influence they receive from all optimistic and pessimistic agents. There is also a new parameter $\frac{\kappa}{1+\mu}$ that is coming from the optimal action choice $x_{i,\infty}$ and scales the influence received by the agent $i$. And, recall that in this society we do not observe a convergence to consensus, instead, every player ends up with an individual asymptotic belief $y_{i,\infty}$. This fact is captured by the last line in iii), such that depending on how far above or below agent $i$’s belief is from $\theta$, she increases or decreases her utility. If $\kappa \to 0$, we will observe a domino effect, such that $y_{i,\infty} \to \theta$, $x_{i,\infty} \to \gamma + \theta$ and, finally, $u_{i,\infty} \to \frac{2^2+2+\theta}{2}$. It is intuitive, since decreasing $\kappa$ means that agents receive more reliable information, and, in that case, the society consists only of regular and Bayesian agents.
We would like to draw your attention on one more fact - the effect that is coming from the stubborn agents in (ii) and polarized agents in (iii). In both cases, the agents subtract the effect they receive directly and the effect they have through their coordinating neighbors. Moreover, there is a third term, a multiplication of both effects, that is added or deducted, depending on \((\theta - y_{j,0}), j \in S\) and \(\left(\sum_{k \in P^+} \Pi_{jk} - \sum_{k \in P^-} \Pi_{jk}\right)\).

**Example 6.** The results of numerical exercises for utility dynamics are given below. All the parameters are kept as in the previous Example.

Utility dynamics

![Utility dynamics graphs](image)

The following graphs explicitly demonstrate the results we have in Proposition 7. Indeed, one observes a full-coordination in asymptotic utilities in the homogeneous society, while it is not the case for the other societies. Stubborn agents bring bias in the utility levels. Note, that agent 7 whose initial belief is higher than the true state \((y_{i,0} > \theta)\), ends up with higher utility, thereby, increasing utilities for the ones who coordinate with her. The opposite thing happens for stubborn agents 8 and 9.

Polarization drives bias in utilities as well. As the analytical result states, there are different asymptotic utilities with the highest ones for optimistic agents, the lowest utilities belong to pessimistic agents and in-between utilities are of regular individuals.

5 Convergence to stable beliefs

In this chapter we would like to investigate how long does it take for the society to converge to stable asymptotic beliefs. In other words, how many time units does a person need to discuss the topic in order to settle down with a stable opinion? What are the main determinants?

We find it reasonable to study the timing, since it is relevant to many practical situations and make the analysis of our model more complete. To start the analysis, we need a notion of a *convergence time*:

**Definition 2.** Convergence time to \(\epsilon > 0\) of a network \(W\) is

\[
CT(\epsilon; W) = \min \{ t : \max_{x \in \Omega} \| W^t(x, \cdot) - \Pi(x, \cdot) \| \leq \epsilon \}\]
Technically, $CT(\epsilon; W)$ gives the minimal number of $t$ that is needed for matrix $W$ to converge to the stable matrix $\Pi$ (see Appendix 7.1). The definition concerns the relationship between time and network properties, and, in particular the second largest eigenvalue of the matrix $W$. This process is well studied in Markov chain literature, where $W$ is seen as a transition matrix. Generally, the more sparsely connected a network is, the larger is the second largest eigenvalue $|\lambda_2|$. For example, a complete network of size $n$ has $|\lambda_2| = \frac{1}{n-1}$, while a cyclic network of size (odd value) $n$ has $|\lambda_2| = 1$. It simply means that complete network needs $t = 1$ to transfer $\lim_{t \to \infty} W^t = \Pi$, while $t$ is larger for cyclic networks.

**Definition 3.** $T_i$ is the time agent $i$ needs in order her opinion $E[y_{i,t}]$ converges to asymptotic belief $y_{i,\infty}$, such that

$$T_i = \min\{t : |E[y_{i,t}] - y_{i,\infty}| \leq \epsilon\}$$

One can think of a repeated communication among the agents. We focus on $E[y_{i,t}]$. Definition 3 means that agent $i$ needs at least $T_i$ periods in order her average opinion becomes $\epsilon$ close to her long-run opinion.

**Proposition 8.** Agent $i \in N$ reaches the stable opinion at time given by:

i) $N = N_r$:

$$T_i = \frac{1}{\varepsilon} \sum_{k=1}^{n} W_{kk} \Pi_{ik} \left[(1 - \theta) \sum_{k=1}^{n} \Pi_{ik} \alpha_{k,0} - \theta \sum_{k=1}^{n} \Pi_{ik} \beta_{k,0}\right] - \varepsilon \left(\sum_{k=1}^{n} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) + \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{kk} W_{ik}^l\right) + CT(\epsilon; W)$$

ii) $N = N_r \cup B \cup S$:

$$T_i = \frac{1}{\varepsilon} \sum_{k \in B} \Pi_{ik} \left[(1 - \theta) \sum_{k \in S \cup B} \Pi_{ik} \alpha_{k,0} - \theta \sum_{k \in S \cup B} \Pi_{ik} \beta_{k,0}\right] - \varepsilon \left(\sum_{k \in S \cup B} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) + \sum_{l=0}^{CT-1} \sum_{k \in N \setminus S} W_{kk} W_{ik}^l\right) + CT(\epsilon; W)$$

---

4In this case, we mean an absolute value of $\lambda_2$. 
iii) $N = N_r \cup P^+ \cup P^-$:

$$T_i = \frac{1}{\epsilon} \left[ \left| (1 - y_{i,\infty}) \sum_{k \in P^-} \Pi_{ik} \alpha_{k,0} - y_{i,\infty} \sum_{k \in P^+} \Pi_{ik} \beta_{k,0} \right| + 
\rho \left( \sum_{k \in P^+} \Pi_{ik} - \sum_{k \in P^-} \Pi_{ik} \right) \sum_{l=0}^{CT-1} \sum_{k \in N \setminus P} \sum_{l=0}^{CT-1} W_{kk} W^l_{ik} + 2 \left( \sum_{k \in P^-} \sum_{l=0}^{CT-1} W_{ik}^l - \sum_{k \in P^+} \sum_{l=0}^{CT-1} W^l_{ik} \right) \right] - \sum_{k \in P^+} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) - \sum_{l=0}^{CT-1} n \sum_{k} W_{kk} W^l_{ik} + CT(\epsilon; W)$$

Since we base the definition of $T_i$ on notions of expected and asymptotic opinions, it is logical that we get the expressions framed by their components. Undoubtedly, the network structure plays a major role since it is presented by: $CT(\epsilon; W)$ that depends on $|\lambda_2|$, $\sum_{l=0}^{CT-1} n \sum_{k=1}^n W_{kk} W^l_{ik}$ that gives us a sum of influences that each agent $k$ has on agent $i$’s behavior till the moment $CT(\epsilon; W)$ multiplied by self-confidence of agent $k$, and, of course, stable matrix $\Pi$. Recall, $\Pi_{ik}$ says how much influence agent $k$ has on agent $i$’s limiting behavior. Apart of that, there are parameters of initial beliefs ($\alpha_{i,0}, \beta_{i,0}$). In contrast with the previous results, initial beliefs help to determine the time of opinion convergence. Note, here the parameters and characteristics of the whole society are taken into account.

Result ii) describes the time for agents in a society with stubborn and Bayesian agents. The expression has the same structure as in i), except the fact that now we focus mainly on the characteristics of stubborn and Bayesian agents. In such a way, only their influences and initial opinions matter for the result. We also draw your attention on the first ratio with influences of all Bayesian agents in the denominator. Hence, increasing their influence on agent $i$, we can decrease $T_i$.

Considering the last expression, recall that with polarized individuals in the society, each agent converges to individual asymptotic belief. That is the reason why we have $y_{i,\infty}$ instead of familiar $\theta$. The second line represents inter-relationship between influences of optimistic and pessimistic individuals on agent $i$, wrapped up by parameter $\kappa$.

In order to analyze the convergence to stable beliefs of the whole society, we introduce the notion of Total time. It is simply the highest time $T_i$ among the agents in the society. In such a way, the whole society holds the stable opinion when every person reaches her stable belief.

**Definition 4.** (Total time) The time needed to converge to stable opinion for the whole society $W$ is

$$T_w = \max \{ T_i \}_{i \in N}$$

**Example 7.** We provide you graphical illustrations of how the values $E[y_{i,t}]$ converge to stable opinions. We use $\epsilon = 0.01$, $\kappa = 0.2$ and $\theta = 0.5$. The characteristics of agents and
the network structure are set as in Example 1. One can conclude that $T_w$ differs notably among the societies.

Convergence of $T_w$ for different societies.

It takes 14 time periods in the homogeneous society for average opinions to converge to long run ones, which in this case is $\theta$. The result for the society with stubborn and Bayesian agents increases up till 1099 time units. Evidently, stubbornness significantly prolongs the learning process. Consider the society with polarized agents, $T_w$ is still higher than the one in the homogeneous society. The reason can be in the heterogeneity in asymptotic opinions, that extends the time to steady state.

<table>
<thead>
<tr>
<th>$N = { N_r, N_r \cup B \cup S, N_r \cup P^+ \cup P^- }$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>$T_w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_r$</td>
<td>13</td>
<td>12</td>
<td>14</td>
<td>11</td>
<td>11</td>
<td>13</td>
<td>13</td>
<td>10</td>
<td>12</td>
<td>12</td>
<td>14</td>
</tr>
<tr>
<td>$N_r \cup B \cup S$</td>
<td>542</td>
<td>52</td>
<td>129</td>
<td>209</td>
<td>1099</td>
<td>730</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>24</td>
<td>1099</td>
</tr>
<tr>
<td>$N_r \cup P^+ \cup P^-$</td>
<td>5</td>
<td>21</td>
<td>15</td>
<td>18</td>
<td>10</td>
<td>18</td>
<td>54</td>
<td>49</td>
<td>102</td>
<td>4</td>
<td>102</td>
</tr>
</tbody>
</table>

Results for $\theta = 0.5$.

The table provides results of $T_i$ for each agent in the society. The last column gives the total time, as the maximal value among the agents. It ensures that all agents have enough time to hold their stable opinions. The conclusion is that: indeed, the presence of stubborn agents significantly affect the timing, strikingly increasing the values of individuals who trust them enormously, while polarized agents have ambiguous effect on $T_i$.

Obtained results allow us to make some general conclusions: First of all, the expressions in Proposition 8 convey a complicated structure of time needed for average opinions to converge to stable ones. Secondly, in contrast to other results, initial opinions play a large role in determining the timing.

**Opinion convergence**

Till now we were analyzing the asymptotic beliefs and long-run decisions of agents, where uncertainty vanishes under particular conditions. However, in order to study time to stable opinions, we have to face uncertainty. We say
\[ \hat{t}_i = \min \{ t : |y_{i,t} - y_{i,\infty}| \leq \varepsilon \} \]

such that \( \hat{t}_i \) is the time when the opinion of agent \( i \) becomes close to her asymptotic belief. By definition, \( y_{i,t} \) is a random value that depends on the signals. Consequently, \( \hat{t}_i \) is also random and we analyze how the average value of \( \hat{t}_i \) and standard deviation differ depending on the social structure. We run numerical exercises in order to get the distribution of \( \hat{t}_i \) for different societies. We work with the same three societies as before (Example 1 - homogeneous society, Example 3 - society with 3 stubborn and 1 Bayesian agents, and as in Example 4 - society with 3 pessimistic and 2 optimistic agents), results are provided for different values of \( \theta = \{0.25, 0.5, 0.8\} \) and we repeat the process 10000 times. To run the exercise we use a familiar concept: \( \hat{t}_w = \max \{ \hat{t}_i \}_{i \in N} \).

\( \hat{t}_w \) is defined as:

\[
xx = \text{abs}(Y[: , t] - Y_{\text{long}}) \\
\text{if all}(xx < e \text{ for } x \text{ in } xx): \text{break}
\]

where \( Y_{\text{long}} \) is a vector of asymptotic opinions and \( e = \varepsilon = 0.01 \). The process stops when the absolute difference between current \( (Y[: , t]) \) and long run \( (Y_{\text{long}}) \) opinion is smaller than \( \varepsilon \) for all the agents in the society. In this case, we say that all agents hold opinion that is at most \( \varepsilon \) far away from their long run beliefs. Conducting the numerical experiment allows us to compare the hard data among the societies.

Below, one can find the results with commonly used statistics: Mean and Std. Dev. for obtained results and graphical illustrations of distributions of \( \hat{t}_w \) with three cases for each society.

<table>
<thead>
<tr>
<th>( N = )</th>
<th>Mean</th>
<th>Std Dev.</th>
<th>Mean</th>
<th>Std Dev.</th>
<th>Mean</th>
<th>Std Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_r )</td>
<td>329.2</td>
<td>326.17</td>
<td>229.62</td>
<td>246.72</td>
<td>311.22</td>
<td>288.87</td>
</tr>
<tr>
<td>( N_r \cup B )</td>
<td>418.67</td>
<td>743.15</td>
<td>512.26</td>
<td>834.14</td>
<td>397.28</td>
<td>636.90</td>
</tr>
<tr>
<td>( N_r \cup B \cup S )</td>
<td>1242.98</td>
<td>1212.79</td>
<td>1071.37</td>
<td>1166.41</td>
<td>3213.27</td>
<td>1724.16</td>
</tr>
<tr>
<td>( N_r \cup P^+ \cup P^- )</td>
<td>577.98</td>
<td>843.19</td>
<td>998.71</td>
<td>1296.27</td>
<td>572.29</td>
<td>766.24</td>
</tr>
</tbody>
</table>

Results for:

\( \theta = 0.25 \) \( \theta = 0.5 \) \( \theta = 0.8 \)

For comparison reasons, we add the results for the society with regular and Bayesian agents, \( N_r \cup B \) with \( |B| = 1 \). According to the table above, the lowest Mean and Std. Dev. are in homogeneous society. It is true for all values of \( \theta \). Despite the fact that Bayesian agents do not affect the asymptotic beliefs of regular agents, the second row states that they do affect the average time to the steady state and its variation. We make the same conclusion for the society with polarized agents. The numbers increase for all values of \( \theta \). However, recall that in this case each agent converges to her own asymptotic belief, \( y_{i,\infty} \). Heterogeneity in initial beliefs, network structure and level of polarization make the society to end up in different steady states. This fact is reflected in average time and
its precision Considering society with stubborn agents, one can state that their impact is significant and negative. Values of mean and standard deviation are the highest among all the results, they are several times higher than in homogeneous society. It is obvious, stubborn behavior slows down the information flow in the network.

In Figure 5 there are graphs that are presented by means of histograms in order to show the frequency distribution of $\hat{t}_w$. First thing one can notice is the difference of distributions between the societies. It is in line with the results in the table that we have seen. In this sense, homogeneous society is more centered with small variance, while the society with polarized agents are characterized with larger dispersion. Middle row graphs have more heavily and widely distributed tails. That is explained by the presence of stubborn agents. Their disinclination to update opinions not only affects social beliefs and actions, but also makes the learning process much longer.

This section is devoted to $T_i$ - time when average opinion reaches the steady one and to $\hat{t}_i$ - time when opinion converges to the asymptotic one. The first result is provided in both analytical and numerical forms, while the latter, due to its complexity, is only presented as a numerical exercise.

Concluding these results, we can say that time depends on many factors such as network structure, initial and asymptotic beliefs. Our results also highlight the importance to study the composition of the society. In this way, the time results show that any heterogeneity (Bayesian, polarized, stubborn agents) in agents’ information processing increases the time compared to the homogeneous society. We would like to point out the drastic effect of stubborn players compared to other types of agents.
Figure 5: Distribution of $\hat{t}_w$

6 Conclusion

In this paper we focus on a model of social learning. Inspired by works of Jadbaia et al. (2012) and Fernandes (2018), we base our model on a combination of rational and boundedly rational learning approaches. More specifically, agents take new arriving information in a Bayesian way and neighbor’s beliefs in a manner of DeGroot model. We call such agents regular ones and study opinion dynamics, decision problem and convergence time to stable opinion when along with regular agents there are either Bayesian or stubborn, or polarized agents.
We start with opinion dynamics and find that when all agents are regular (such that they update their beliefs in mixed way), there is successful information aggregation. It occurs independently of the social structure, their initial beliefs and the state of the world. Our results show that the presence of Bayesian individuals does not influence asymptotic opinions, while stubborn behavior precludes convergence, generating persistent disagreement in the society. Our interesting finding is that, at least one Bayesian individual can overcome this fluctuation that stubborn agents cause. In other words, Bayesian and regular agents reveal the true state, while stubborn individuals keep their initial opinions. This result gives an insight on how to provide the successful learning despite the presence of stubborn agents. The society with polarized agents has convergence of opinions, but faces misinformation due to the bias of signals of polarized agents. Unlike for previous results, the structure of trust relationships is one of the main determinants for asymptotic beliefs.

Formed beliefs inevitably influence decisions that agents take. Indeed, in our framework, one of the main components of optimal action is a social belief. Along with it, actions are affected by coordination and individualistic parameters. In homogeneous society, thanks to successful learning, there is full coordination in optimal actions and utilities. Our results show that even if Bayesian agents can preclude the influence of stubborn agents and provide opinion convergence to the true state, it does not work when agents face the action problem. In this case, the number of stubborn players and agents’ willingness to coordinate with them matters. Consequently, actions are affected and shifted from the one in homogeneous society (without stubborn agents). Similar situation we observe for the society with polarized agents. Number of optimists and pessimists, coordination with them and trust values people put on them, all this determine long-run actions and utilities.

In addition, we provide the analysis of time to steady opinion. Depending on the social composition, initial and asymptotic beliefs, network structure and true state of the world determine the time that average opinion reaches a stable one. Both analytical and empirical results show that heterogeneity in constitution of the society increases the time to converge to stable opinion. Note, there is an increase in average time and variance in the society with Bayesian agents compared to the homogeneous society. That means, unlike the previous results, there is a negative effect, despite the society reaches the true state. At the same time, we would like to highlight the impact of stubborn agents who prolong the learning process that in some cases might hurt badly (think of vaccination example). Our results point out the importance of the composition of the society.

In the future perspectives, it would be interesting to consider key players in the society: the players who are crucial in reducing the convergence time. Another question could be to consider the optimal placement of stubborn agents in order to maximally reduce negative effect.
Acknowledgements

I would like to thank Agnieszka Rusinowska and Vincent Vannetelbosch for a great supervision and support, Ana Mauleon, Michel Grabisch, Berno Buechel, René van den Brink and Johannes Johnen for helpful comments and suggestions. I acknowledge the support by the Project ExSIDE. This work has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Grant Agreement No. 721846, “Expectations and Social Influence Dynamics in Economics (ExSIDE)".

7 Appendix

7.1 Background

Nonnegative matrices

A matrix $W$ is primitive if $W^t > 0$ for some integer $t$, i.e. each entry is positive. It is irreducible if for every $i, j \in N$ there exists an integer $m(i, j)$ such that $W_{ij}^{(m(i,j))} > 0$. $W$ is aperiodic if the greatest common divisor of all the directed cycle lengths is 1. $W$ is primitive iff $W$ is irreducible and aperiodic.

Let us consider a row-stochastic matrix $W$. The Perron-Frobenius Theorem implies if $W$ is primitive then

$$\lim_{t \to \infty} W^t = \Pi = 1 \cdot \pi^T \tag{7.1}$$

where $\pi^T$ is a unique left eigenvector (i.e. $\pi^T W = \pi^T$) such that $\pi^T \cdot 1 = 1$ and $1$ is a vector of ones. In another words, the matrices $W^t$ approach a limiting matrix $\Pi$, where each row of $\Pi$ is equal to the stationary $\pi^T$.

A state in a Markov chain is absorbing if and only if the row corresponding to the state has 1 on the main diagonal and 0’s everywhere else. If $W$ is the matrix of an absorbing Markov chain, then there is a limiting matrix $\lim_{t \to \infty} W^t = \Pi$, where

$$W = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix} \quad \Pi = \begin{bmatrix} 0 \\ [I - Q]^{-1}R \\ I \end{bmatrix} \tag{7.2}$$

Here $I$ is an $r \times r$ identity matrix with $r$ as a number of absorbing states, $0$ is an $r \times (n-r)$ zero matrix, $Q$ is $(n-r) \times (n-r)$ nonzero matrix and $R$ is $(n-r) \times r$ matrix.

We use these results extensively in our Propositions. Equation 7.1 describes the long run trust weights agents assign to each other. It is true for homogeneous society, where interaction matrix is primitive. However, for societies with stubborn and Bayesian agents we need Equation 7.2, where they present absorbing states in the sense of Markov chain theory. The matrix $Q$ describes the weights that social agents assign to each other, while $R$ shows the weights the social agents put on the opinions of stubborn and Bayesian
agents. Consequently, in the long run the society trusts only to them.

**Time to consensus**
Definition 2 is in line with Golub and Jackson (2012) and is based on mixing times of Markov processes [see Montenegro and Tetali (2006)]. It is evident that beyond the consensus time, \( t > CT \), the primitive matrix \( W \) satisfies (7.1).

**Lemma 1.** (Golub and Jackson (2012), Lemma 2). Let \( W \) be connected, \( \lambda_2(W) \) be the second largest eigenvalue in magnitude of \( W \), and \( w = \min_i w \) be the minimum weight. If \( \lambda_2 \neq 0 \), then for any \( 0 < \epsilon \leq 1 \):

\[
\left\lfloor \log\left(\frac{1}{2\epsilon}\right) - \log\left(\frac{1}{\lambda_2(W)}\right) \right\rfloor \leq CT(\epsilon; W) \leq \left\lceil \log\left(\frac{1}{\epsilon}\right) \log\left(\frac{1}{|\lambda_2(W)|}\right) \right\rceil
\]

Since we take into account the social interaction via the adjacency matrix, the following expression is useful to determine the time to steady state for matrix \( W \). In the paper we assume \( CT(\epsilon; W) = \left\lceil \log\left(\frac{1}{\epsilon}\right) \log\left(\frac{1}{|\lambda_2(W)|}\right) \right\rceil \), expecting the worst scenario.

**Beta-Bernoulli model**
At time \( t \), the belief of agent \( i \) is represented by the Beta probability distribution with parameters \( \alpha_{i,t} \) and \( \beta_{i,t} \)

\[
f_{i,t}(\theta|\alpha_{i,t}, \beta_{i,t}) =\begin{cases} \frac{1}{B(\alpha_{i,t}, \beta_{i,t})}\theta^{\alpha_{i,t}-1}(1-\theta)^{\beta_{i,t}-1}, & \text{for } 0 < \theta < 1 \\ 0, & \text{otherwise} \end{cases}
\]

where the term \( \frac{1}{B(\alpha_{i,t}, \beta_{i,t})} \) is present to act as a normalizing constant so that the area under the total probability actually sums to 1. In this sense,

\[
f_{i,t}(\theta|\alpha_{i,t}, \beta_{i,t}) \propto \theta^{\alpha_{i,t}-1}(1-\theta)^{\beta_{i,t}-1} \tag{7.3}
\]

We assume that signals are binary, 1 and 0, and agents need to learn what is the true value of \( \theta \). Hence, we use Bernoulli likelihood function:

\[
l_i(s_{t+1}|\theta) = \theta^{s_{t+1}^{(1)}}(1-\theta)^{s_{t+1}^{(0)}}
\]

In Bayes’ rule the posterior distribution is proportional to the product of the prior distribution and the likelihood function:

\[
f_{i,t+1}(\theta|s_{t+1}) \propto l_i(s_{t+1}|\theta)f_{i,t}(\theta) \propto \theta^{\alpha_{i,t}+s_{t+1}^{(1)}}(1-\theta)^{\beta_{i,t}+s_{t+1}^{(0)}} - 1.
\]

Therefore, the posterior distribution is

\[
f_{i,t+1}(\theta|s_{t+1}) =\begin{cases} \frac{1}{B(\alpha_{i,t+1}, \beta_{i,t+1})}\theta^{\alpha_{i,t+1}-1}(1-\theta)^{\beta_{i,t+1}-1}, & \text{for } 0 < \theta < 1 \\ 0, & \text{otherwise} \end{cases}
\]
where $\alpha_{i,t+1} = \alpha_{i,t} + s_{i,t+1}^{(1)}$ and $\beta_{i,t+1} = \beta_{i,t} + s_{i,t+1}^{(0)}$.

7.2 Proofs

We recall the formulation of opinion of agent $i$ in (3.1) and (3.3):

\[ y_{i,t} = \frac{\alpha_{i,t}}{\alpha_{i,t} + \beta_{i,t}} \]
\[ \alpha_{i,t+1} = W_i[\alpha_{i,t} + s_{i,t+1}^{(1)}] + \sum_{k \neq i} W_{ik} \alpha_{k,t} \]
\[ \beta_{i,t+1} = W_i[\beta_{i,t} + s_{i,t+1}^{(0)}] + \sum_{k \neq i} W_{ik} \beta_{k,t} \]

Let $W^d = diag(W)$, such that we keep only elements on the diagonal and take off-diagonal ones as zero. Then $\alpha_t$ can be written in the matrix form:

\[ \alpha_t = W_t \alpha_0 + \sum_{l=0}^{t-1} W^d s_{t-l}^{1} \]

Analogously:

\[ \beta_t = W_t \beta_0 + \sum_{l=0}^{t-1} W^d s_{t-l}^{0} \]

Then the belief of agent $i$:

\[ \alpha_{i,t} = \sum_{k=1}^{n} W_{ik} \alpha_{k,0} + \sum_{l=0}^{t-1} \sum_{k=1}^{n} W_{kk} W^d W_{lk} s_{k,t-l}^{1} \]
\[ \beta_{i,t} = \sum_{k=1}^{n} W_{ik} \beta_{k,0} + \sum_{l=0}^{t-1} \sum_{k=1}^{n} W_{kk} W^d W_{lk} s_{k,t-l}^{0} \]

(7.4)

Then the belief of agent $i$:

\[ \alpha_{i,t} + \beta_{i,t} = \sum_{k=1}^{n} W_{ik} (\alpha_{k,0} + \beta_{k,0}) + \sum_{l=0}^{t-1} \sum_{k=1}^{n} W_{kk} W^d W_{lk} (s_{k,t-l}^{1} + s_{k,t-l}^{0}) \]

Proof. Proposition 1

Note that the analytical result of Proposition 1 is based on the technique provided by Fernandes (2018). It is a particular case of his Proposition 5, where we are not taking into account the confirmatory bias.

According to Equation 7.4, the opinion of agent $i$ at time $t$ can be written as:

\[ y_{i,t} = \frac{\sum_{k=1}^{n} W_{ik} \alpha_{k,0} + \sum_{l=0}^{t-1} \sum_{k=1}^{n} W_{kk} W^d W_{lk} s_{k,t-l}^{1}}{\sum_{k=1}^{n} W_{ik} (\alpha_{k,0} + \beta_{k,0}) + \sum_{l=0}^{t-1} \sum_{k=1}^{n} W_{kk} W^d W_{lk} (s_{k,t-l}^{1} + s_{k,t-l}^{0})} \]
In the long run, the opinion will take a form:

\[
\lim_{t \to \infty} y_{i,t} = \lim_{t \to \infty} \frac{\sum_{k=1}^{n} W_{ik}^t \alpha_{k,0} + \sum_{l=0}^{t-1} \sum_{k=1}^{n} W_{kk} W_{ik}^l s_{k,t-l}^1}{\sum_{k=1}^{n} W_{ik}^t (\alpha_{k,0} + \beta_{k,0}) + \sum_{l=0}^{t-1} \sum_{k=1}^{n} W_{kk} W_{ik}^l (s_{k,t-l}^1 + s_{k,t-l}^0)}
\]

(7.5)

According to Equation 7.1, the matrix \( W \) is primitive and therefore \( \lim_{t \to \infty} W^t = \Pi \).

\[
\lim_{t \to \infty} \frac{\sum_{k=1}^{n} \Pi_{ik} \alpha_{k,0} + \sum_{l=0}^{t-1} \sum_{k=1}^{n} W_{kk} W_{ik}^l s_{k,t-l}^1 + \sum_{l=0}^{t-1} \sum_{k=1}^{n} W_{kk} \Pi_{ik} s_{k,t-l}^1}{\sum_{k=1}^{n} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) + \sum_{l=0}^{t-1} \sum_{k=1}^{n} W_{kk} W_{ik}^l (s_{k,t-l}^1 + s_{k,t-l}^0)}
\]

Recall that \( CT(\epsilon, W) \) is time when the distance between \( W \) and \( \Pi \) is smaller than error term \( \epsilon \). Hence, from \( CT(\epsilon, W) \) \(^5\) and onwards we can substitute \( W^t \) by \( \Pi \):

\[
\lim_{t \to \infty} \frac{\sum_{k=1}^{n} \Pi_{ik} \alpha_{k,0} + \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{kk} W_{ik}^l s_{k,t-l}^1 + \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{kk} \Pi_{ik} s_{k,t-l}^1}{\sum_{k=1}^{n} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) + \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{kk} W_{ik}^l (s_{k,t-l}^1 + s_{k,t-l}^0) + \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{kk} \Pi_{ik} (s_{k,t-l}^1 + s_{k,t-l}^0)}
\]

\[
= \lim_{t \to \infty} \frac{\frac{1}{t} \sum_{k=1}^{n} \Pi_{ik} \alpha_{k,0} + \frac{1}{t} \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{kk} W_{ik}^l s_{k,t-l}^1 + \frac{1}{t} \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{kk} \Pi_{ik} s_{k,t-l}^1}{\frac{1}{t} \sum_{k=1}^{n} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) + \frac{1}{t} \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{kk} W_{ik}^l (s_{k,t-l}^1 + s_{k,t-l}^0) + \frac{1}{t} \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{kk} \Pi_{ik} (s_{k,t-l}^1 + s_{k,t-l}^0)}
\]

\[
= \lim_{t \to \infty} \frac{\frac{1}{t} \sum_{l=0}^{t-1} \sum_{k=1}^{n} W_{kk} \Pi_{ik} s_{k,t-l}^1}{\frac{1}{t} \sum_{l=0}^{t-1} \sum_{k=1}^{n} W_{kk} \Pi_{ik} (s_{k,t-l}^1 + s_{k,t-l}^0)}
\]

\[
= \lim_{t \to \infty} \frac{\sum_{k=1}^{n} W_{kk} \Pi_{ik} \frac{t - CT}{t} \sum_{l=0}^{t-1} s_{k,t-l}^1}{\sum_{k=1}^{n} W_{kk} \Pi_{ik} \frac{t - CT}{t} \sum_{l=0}^{t-1} (s_{k,t-l}^1 + s_{k,t-l}^0)}
\]

\(^5\)In order to avoid complexity in expressions, we will refer to this term as \( CT \).
\[\lim_{t \to \infty} \frac{\sum_{k=1}^{n} W_{kk} \Pi_{ik} \frac{1}{t - CT} \sum_{l=CT}^{t-1} s_{k,t-l}^1}{\sum_{k=1}^{n} W_{kk} \Pi_{ik} \frac{1}{t - CT} \sum_{l=CT}^{t-1} (s_{k,t-l}^1 + s_{k,t-l}^0)} = \lim_{t \to \infty} \sum_{k=1}^{n} W_{kk} \Pi_{ik} \mathbb{E}[\mathbb{1}\{s_{k,t} = 1\}]
\]
\[= \sum_{k=1}^{n} W_{kk} \Pi_{ik} (\mathbb{E}[\mathbb{1}\{s_{k,t} = 1\} + \mathbb{1}\{s_{k,t} = 0\}])
\]
\[= \sum_{k=1}^{n} W_{kk} \Pi_{ik} \theta
\]
\[= \sum_{k=1}^{n} W_{kk} \Pi_{ik}((\theta + 1 - \theta)
\]
\[= \theta
\]

\[\text{Proof. Proposition 2}
\]
We continue from Equation 7.5. Recall that presence of Bayesian agents means that matrix \(W\) is not primitive any more. Instead, Bayesian agents are absorbing states in terms of Markov chain theory. That is why we refer to Equation 7.2. Therefore, we get:

\[\lim_{t \to \infty} y_{i,t} = \lim_{t \to \infty} \frac{\sum_{k \in B} \Pi_{ik} \alpha_{k,o} + \sum_{l=0}^{CT-1} \sum_{k=1}^{n} w_{kk} W_{ik}^l s_{k,t-l}^1 + \sum_{l=CT}^{t-1} \sum_{k \in B} w_{kk} \Pi_{ik} s_{k,t-l}^1}{\sum_{k \in B} \Pi_{ik} (\alpha_{k,o} + \beta_{k,0}) + \sum_{l=0}^{CT-1} \sum_{k=1}^{n} w_{kk} W_{ik}^l (s_{k,t-l}^1 + s_{k,t-l}^0) + \sum_{l=CT}^{t-1} \sum_{k \in B} w_{kk} \Pi_{ik} (s_{k,t-l}^1 + s_{k,t-l}^0)}
\]

Following the steps of previous proof, we get:

\[\sum_{k \in B} w_{kk} \Pi_{ik} \theta
\]
\[= \sum_{k \in B} w_{kk} \Pi_{ik}((\theta + 1 - \theta)
\]
\[= \theta
\]

\[\text{Proof. Proposition 3}
\]
Let us present Equation 7.5 in extended way:
Equation 7.6 with presence of stubborn and Bayesian agents becomes:

\[
\begin{align*}
\lim_{t \to \infty} \frac{\sum_{k=1}^{n} \Pi_{ik} \alpha_{k,0} + \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{kk} W^l_{ik} s_{k,t-l}^1 + \sum_{l=CT}^{t-1} \sum_{k=1}^{n} W_{kk} \Pi_{ik} s_{k,t-l}^1}{\sum_{k=1}^{n} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) + \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{kk} W^l_{ik} (s_{k,t-l}^1 + s_{k,t-l}^0) + \sum_{l=CT}^{t-1} \sum_{k=1}^{n} W_{kk} \Pi_{ik} (s_{k,t-l}^1 + s_{k,t-l}^0)}
\end{align*}
\]

(7.6)

As long as there are stubborn agents in the society who weight only their own opinion, matrix \(W\) satisfies Equation 7.2. It means that \(\Pi\) has positive entries only in columns of stubborn players, such that \(\Pi_{ij} > 0\) if \(j \in S\) and 0 otherwise.

\[
\begin{align*}
\lim_{t \to \infty} \frac{\sum_{j \in S} \Pi_{ij} \alpha_{j,0} + \sum_{l=0}^{CT-1} \sum_{k \in N\setminus S} W_{kk} W^l_{ik} s_{k,t-l}^1}{\sum_{j \in S} \Pi_{ij} (\alpha_{j,0} + \beta_{j,0}) + \sum_{l=0}^{CT-1} \sum_{k \in N\setminus S} W_{kk} W^l_{ik}}
\end{align*}
\]

Stubborn agents do not receive any signals \((s_{j,t}^1 = s_{j,t}^0 = 0)\), then for \(\theta \in (0, 1)\) we get for \(i \neq j\):

\[
\begin{align*}
\lim_{t \to \infty} \frac{\sum_{j \in S} \Pi_{ij} \alpha_{j,0} + \sum_{l=0}^{CT-1} \sum_{k \in N\setminus S} W_{kk} W^l_{ik} s_{k,t-l}^1}{\sum_{j \in S} \Pi_{ij} (\alpha_{j,0} + \beta_{j,0}) + \sum_{l=0}^{CT-1} \sum_{k \in N\setminus S} W_{kk} W^l_{ik}}
\end{align*}
\]

Proof. Proposition 4

Equation 7.6 with presence of stubborn and Bayesian agents becomes:

\[
\begin{align*}
\lim_{t \to \infty} \frac{\sum_{k \in S \cup B} \Pi_{ik} \alpha_{k,0} + \sum_{l=0}^{CT-1} \sum_{k \in N\setminus S} W_{kk} W^l_{ik} s_{k,t-l}^1 + \sum_{l=CT}^{t-1} \sum_{k \in B} \Pi_{ik} W_{kk} s_{k,t-l}^1}{\sum_{k \in S \cup B} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) + \sum_{l=0}^{CT-1} \sum_{k \in N\setminus S} W_{kk} W^l_{ik} s_{k,t-l}^1 + \sum_{l=CT}^{t-1} \sum_{k \in B} \Pi_{ik} W_{kk} (s_{k,t-l}^1 + s_{k,t-l}^0)}
\end{align*}
\]

\[
\begin{align*}
\lim_{t \to \infty} \frac{\frac{1}{l} \sum_{k \in S \cup B} \Pi_{ik} \alpha_{k,0} + \frac{1}{l} \sum_{l=0}^{CT-1} \sum_{k \in N\setminus S} W_{kk} W^l_{ik} s_{k,t-l}^1 + \sum_{l=CT}^{t-1} \sum_{k \in B} \Pi_{ik} W_{kk} \frac{1}{l} \sum_{l=CT}^{t-1} s_{k,t-l}^1}{\frac{1}{l} \sum_{k \in S \cup B} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) + \frac{1}{l} \sum_{l=0}^{CT-1} \sum_{k \in N\setminus S} W_{kk} W^l_{ik} (s_{k,t-l}^1 + s_{k,t-l}^0) + \sum_{l=CT}^{t-1} \sum_{k \in B} \Pi_{ik} W_{kk} \frac{1}{l} \sum_{l=CT}^{t-1} (s_{k,t-l}^1 + s_{k,t-l}^0)}
\end{align*}
\]

35
Polarized agents modify the expression \( \lim_{t \to \infty} \). Proof.

Proposition 5

\[ \lim_{t \to \infty} \sum_{k \in B} \frac{1}{t} \sum_{l=CT}^{t-1} s_{k,t-l} = \lim_{t \to \infty} \sum_{k \in B} \frac{1}{t} \sum_{l=CT}^{t-1} (s_{k,t-l} + s_{k,t-l}^0) \]

\[ = \lim_{t \to \infty} \sum_{k \in B} \frac{t - CT}{t} \frac{1}{t - CT} \sum_{l=CT}^{t-1} (s_{k,t-l} + s_{k,t-l}^0) \]

\[ = \lim_{t \to \infty} \sum_{k \in B} \frac{1}{t - CT} \sum_{l=CT}^{t-1} s_{k,t-l} \]

\[ = \lim_{t \to \infty} \sum_{k \in B} \Pi_{ik} \mathbb{E}[\mathbb{1}\{s_{k,t} = 1\}] = \theta \]

\[ = \sum_{k \in B} \Pi_{ik}(\mathbb{E}[\mathbb{1}\{s_{k,t} = 1\}] + \mathbb{1}\{s_{k,t} = 0\}) \]

\[
Proof. \text{ Proposition 5}
\]

Polarized agents modify the expression \( \lim_{t \to \infty} y_{i,t} \) in this way:

\[ \sum_{k \in P} \Pi_{ik} \alpha_{k,0} + \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{k,k} W_{ik}^l s_{k,t-l} + \sum_{l=CT}^{t-1} \sum_{k \in P} \Pi_{ik} s_{k,t-l} \]

\[ = \lim_{t \to \infty} \sum_{k \in P} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) + \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{k,k} W_{ik}^l (s_{k,t-l} + s_{k,t-l}^0) + \sum_{l=CT}^{t-1} \sum_{k \in P} \Pi_{ik} (s_{k,t-l} + s_{k,t-l}^0) \]

\[ = \lim_{t \to \infty} \frac{1}{t} \sum_{k \in P} \Pi_{ik} \alpha_{k,0} + \frac{1}{t} \sum_{l=0}^{CT-1} \sum_{k=1}^{n} w_{k,k} W_{ik}^l s_{k,t-l} + \frac{1}{t} \sum_{l=CT}^{t-1} \sum_{k \in P} \Pi_{ik} s_{k,t-l} \]

\[ = \lim_{t \to \infty} \frac{1}{t} \sum_{k \in P} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) + \frac{1}{t} \sum_{l=0}^{CT-1} \sum_{k=1}^{n} w_{k,k} W_{ik}^l (s_{k,t-l} + s_{k,t-l}^0) + \frac{1}{t} \sum_{l=CT}^{t-1} \sum_{k \in P} \Pi_{ik} (s_{k,t-l} + s_{k,t-l}^0) \]

\[ = \lim_{t \to \infty} \sum_{k \in P} \Pi_{ik} \frac{1}{t} \sum_{l=CT}^{t-1} s_{k,t-l} \]

Recall that \( \mathbb{E}[\mathbb{1}\{s_{k,t} = 1\}] = \theta + \kappa \) for \( k \in P^+ \) and \( \mathbb{E}[\mathbb{1}\{s_{k,t} = 1\}] = \theta - \kappa \) for \( k \in P^- \):
\[
\sum_{k \in P^+} \Pi_{ik} \mathbb{E}[1 \{s_{k,t} = 1\}] + \sum_{k \in P^-} \Pi_{ik} \mathbb{E}[1 \{s_{k,t} = 1\}]
\]
\[
\sum_{k \in P} \Pi_{ik} (\mathbb{E}[1 \{s_{k,t} = 1\}] + 1 \{s_{k,t} = 0\})
\]
\[
= \theta - \kappa (\sum_{k \in P^-} \Pi_{ik} - \sum_{k \in P^+} \Pi_{ik})
\]

Proof. Proposition 6
i) In the long run the agents in homogeneous society converge to true state \(\theta\), such as:
\[
x^*_i = \left[I - CR\right]^{-1} B(\gamma + \theta)
\]
Recall that \(D = \left[I - CR\right]^{-1}\), \(x^*_{i,\infty}\) can be expressed as:
\[
x^*_{i,\infty} = D_{i,1} \frac{\gamma + \theta}{1 + \mu} + D_{i,2} \frac{\gamma + \theta}{1 + \mu} + \cdots + D_{i,n} \frac{\gamma + \theta}{1 + \mu}
\]
\[
= \frac{\gamma + \theta}{1 + \mu} \sum_{k} D_{ik}
\]
For an invertible square matrix \(\left[I - CR\right]\), if the sum of the elements of each row of the matrix is \(k\), then the sum of the elements in each row of the inverse matrix is \(\frac{1}{k}\). In our case, \(k = \frac{1}{1+\mu}\) and \(\sum_{k} D_{ik} = 1 + \mu\), therefore
\[
x^*_{i,\infty} = \gamma + \theta
\]
ii) With the presence of stubborn agents and at least one Bayesian, there is successful learning of all agents \(i \in N \setminus S\). It modifies the expression, where \(S = \{j_1, j_2, ..., j_s\}\):
\[
x^*_{i,\infty} = D_{i,1} \frac{\gamma + \theta}{1 + \mu} + D_{i,2} \frac{\gamma + \theta}{1 + \mu} + \cdots + D_{i,j_1} \frac{\gamma + y_{j_1,0}}{1 + \mu} + \cdots + D_{i,j_s} \frac{\gamma + y_{j_s,0}}{1 + \mu} + D_{i,n} \frac{\gamma + \theta}{1 + \mu}
\]
\[
= \frac{\gamma + \theta}{1 + \mu} \left(1 + \mu - \sum_{j \in S} D_{ij}\right) + \sum_{j \in S} D_{ij} \frac{\gamma + y_{j,0}}{1 + \mu}
\]
\[
= \gamma + \theta - \sum_{j \in S} D_{ij} \frac{\theta - y_{j,0}}{1 + \mu}
\]
iii) Polarized agents prevent the society from revealing true state:

\[ x_{i,\infty}^* = D_{i,1} \frac{\gamma + \theta + \kappa(\sum_{k \in P^+} \Pi_{ik} - \sum_{k \in P^-} \Pi_{ik})}{1 + \mu} + \cdots + D_{i,k} \frac{\gamma + \theta + \kappa}{1 + \mu} + \cdots + D_{i,j} \frac{\gamma + \theta - \kappa}{1 + \mu} \]

\[ = \sum_{j \in N \setminus S} D_{ij} \frac{\gamma + \theta}{1 + \mu} + \sum_{j \in N \setminus S} D_{ij} \frac{\kappa(\sum_{k \in P^+} \Pi_{jk} - \sum_{k \in P^-} \Pi_{jk})}{1 + \mu} + \sum_{j \in P^+} D_{ij} \frac{\gamma + \theta + \kappa}{1 + \mu} + \sum_{j \in P^-} D_{ij} \frac{\gamma + \theta - \kappa}{1 + \mu} \]

Given that \( \sum_{j \in N \setminus S} D_{ij} + \sum_{j \in P^+} D_{ij} + \sum_{j \in P^-} D_{ij} = 1 + \mu \), we get the following expression:

\[ = \gamma + \theta + \frac{\kappa}{1 + \mu} \sum_{j \in N} D_{ij}(\sum_{k \in P^+} \Pi_{jk} - \sum_{k \in P^-} \Pi_{jk}) \]

\( \square \)

**Proof. Proposition 7**

Taking into account Proposition 6, we substitute the results in

\[ u_i(x^*_\infty) = x^*_{i,\infty} \left( \gamma + y_{i,\infty} + \mu \sum_j r_{ij} x^*_{j,\infty} \right) - x^2_{i,\infty} \left( \frac{1 + \mu}{2} \right) - \frac{1}{2} \left( y_{i,\infty}^2 + \mu(\sum_j r_{ij} x^*_{j,\infty})^2 \right) \]

\( i \)

\[ u_i(x^*_\infty) = (\gamma + \theta) (\gamma + \theta + \mu (\gamma + \theta)) - \frac{(\gamma + \theta)^2 (1 + \mu)}{2} - \frac{1}{2} \left( \theta^2 + \mu (\gamma + \theta)^2 \right) \]

\[ = \frac{\gamma^2 + 2\gamma \theta}{2} \]

\( ii \) Recall that in the second case we take into account stubborn opinions in the society:

\[ u_i(x^*_\infty) = \left( \gamma + \theta - \sum_{j \in S} D_{ij} \left( \frac{\gamma + \theta - y_{j,0}}{1 + \mu} \right) \right) \left( \gamma + \theta + \mu \sum_{k \in R_i} r_{ik} (\gamma + \theta - \sum_{j \in S} D_{kj} \left( \frac{\theta - y_{j,0}}{1 + \mu} \right) \right) \]

\[ - \left( \gamma + \theta - \sum_{j \in S} D_{ij} \left( \frac{\gamma + \theta - y_{j,0}}{1 + \mu} \right) \right)^2 \left( \frac{1 + \mu}{2} \right) - \frac{1}{2} \left( \theta^2 + \mu \left( \sum_{k \in R_i} \left( \gamma + \theta - \sum_{j \in S} D_{kj} \left( \frac{\theta - y_{j,0}}{1 + \mu} \right) \right) \right)^2 \right) \]

\[ = \frac{\gamma^2 + 2\gamma \theta}{2} - \frac{\mu}{2r_i^2} \left( \sum_{k \in R_i} \sum_{j \in S} D_{kj} \left( \frac{\theta - y_{j,0}}{1 + \mu} \right)^2 \right) - \frac{1 + \mu}{2} \left( \sum_{j \in S} D_{ij} \left( \frac{\theta - y_{j,0}}{1 + \mu} \right)^2 \right) + \]

\[ + \frac{\mu}{r_i} \left( \sum_{k \in R_i} \sum_{j \in S} D_{kj} \left( \frac{\theta - y_{j,0}}{1 + \mu} \right) \right) \left( \sum_{j \in S} D_{ij} \left( \frac{\theta - y_{j,0}}{1 + \mu} \right) \right) \]

Note that utility for stubborn agents is calculated with \( y_{i,\infty} = y_{i,0} \) for \( i \in S \).
iii) There is no consensus in the society. Recall that \( y_{t,\infty} = \theta - \kappa \left( \sum_{k \in P^-} \Pi_{ik} - \sum_{k \in P^+} \Pi_{ik} \right) \):

\[
u_t(x^*_\infty) = \left( \gamma + \theta + \frac{\kappa}{1 + \mu} \sum_{j \in N} D_{ij} \left( \sum_{k \in P^+} \Pi_{jk} - \sum_{k \in P^-} \Pi_{jk} \right) \right) \left( \gamma + y_{t,\infty} + \mu \sum_{j \in R_t} r_{ij} (\gamma + \theta + \frac{\kappa}{1 + \mu} \sum_{l \in N} D_{jl} \left( \sum_{k \in P^+} \Pi_{lk} - \sum_{k \in P^-} \Pi_{lk} \right) \right)^2
\]

\[
\left( \frac{1 + \mu}{2} \right) - \frac{1}{2} \left( y_{t,\infty} + \mu \sum_{j \in R_t} r_{ij} (\gamma + \theta + \frac{\kappa}{1 + \mu} \sum_{l \in N} D_{jl} \left( \sum_{k \in P^+} \Pi_{lk} - \sum_{k \in P^-} \Pi_{lk} \right) \right)^2
\]

\[
= \frac{\gamma^2 + 2\mu y_{t,\infty}}{2} \]

\[
- \frac{\kappa^2}{2(1 + \mu)} \left( \sum_{j \in N} D_{ij} \left( \sum_{k \in P^+} \Pi_{jk} - \sum_{k \in P^-} \Pi_{jk} \right) \right)^2 \]

\[
- 2(1 + \mu)^2 \left( \frac{1}{r_i} \sum_{l \in N} D_{jl} \left( \sum_{k \in P^+} \Pi_{lk} - \sum_{k \in P^-} \Pi_{lk} \right) \right)^2
\]

\[
+ \frac{\mu^2}{(1 + \mu)^2} \left( \sum_{j \in N} D_{ij} \left( \sum_{k \in P^+} \Pi_{jk} - \sum_{k \in P^-} \Pi_{jk} \right) \right) \left( \sum_{j \in N} D_{ij} \left( \sum_{k \in P^+} \Pi_{jk} - \sum_{k \in P^-} \Pi_{jk} \right) \right) - \frac{(y_{t,\infty} - \theta)^2}{2} + \frac{\kappa(y_{t,\infty} - \theta)}{1 + \mu} \left( \sum_{j \in N} D_{ij} \left( \sum_{k \in P^+} \Pi_{jk} - \sum_{k \in P^-} \Pi_{jk} \right) \right)
\]

\[

\text{Proof. Proposition 8}
\]

i) According to Proposition 1, \( y_{t,\infty} = \theta \), then it follows:

\[
\begin{align*}
\alpha_{t,\infty} &= \sum_{k=1}^{n} W_{ik} \alpha_{k,0} + \sum_{l=1}^{t-1} \sum_{k=1}^{n} W_{kk} W_{ik} s_{k,t-l} \\
\beta_{t,\infty} &= \sum_{k=1}^{n} W_{ik} \beta_{k,0} + \sum_{l=1}^{t-1} \sum_{k=1}^{n} W_{kk} W_{ik} s_{k,t-l} \\
|\mathbb{E}[y_{t,\infty}] - y_{t,\infty}| &\leq \varepsilon
\end{align*}
\]

We have two possibilities:

\[
\begin{cases}
|\mathbb{E}[y_{t,\infty}] - y_{\infty}| = \mathbb{E}[y_{t,\infty}] - y_{\infty} & \text{if } \mathbb{E}[y_{t,\infty}] - y_{\infty} \geq 0 \text{ or } \mathbb{E}[y_{t,\infty}] \geq y_{\infty} \text{ or } \mathbb{E}[y_{t,\infty}] \geq \theta \\
|\mathbb{E}[y_{t,\infty}] - y_{\infty}| = -(\mathbb{E}[y_{t,\infty}] - y_{\infty}) & \text{if } \mathbb{E}[y_{t,\infty}] - y_{\infty} < 0 \text{ or } \mathbb{E}[y_{t,\infty}] < y_{\infty} \text{ or } \mathbb{E}[y_{t,\infty}] < \theta
\end{cases}
\]

We capture the situations where the expected opinion of an agent can be higher or lower than true state, but, eventually, no matter from which side, the opinion converges to the truth.

1. Case: \( |\mathbb{E}[y_{t,\infty}] - y_{\infty}| = \mathbb{E}[y_{t,\infty}] - y_{\infty} \).

\[
\sum_{k=1}^{n} \Pi_{ik} \alpha_{k,0} + \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{kk} W_{ik} \mathbb{E}[s_{k,t-l}] + \sum_{k=1}^{n} W_{kk} \Pi_{ik} \mathbb{E}[s_{k,t-l}] - \theta \leq \varepsilon
\]

\[
\sum_{k=1}^{n} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) + \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{kk} W_{ik} (\mathbb{E}[s_{k,t-l}] + \mathbb{E}[s_{k,t-l}^0]) + \sum_{k=1}^{n} W_{kk} \Pi_{ik} \sum_{l=CT}^{t-1} (\mathbb{E}[s_{k,t-l}] + \mathbb{E}[s_{k,t-l}^0])
\]
\[
\sum_{k=1}^{n} \Pi_{ik} \alpha_{k,0} + \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{kk} W_{lk}^l \theta + (t - CT) \theta \sum_{k=1}^{n} W_{kk} \Pi_{ik} - \theta \sum_{k=1}^{n} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) \leq \varepsilon
\]

\[
\sum_{k=1}^{n} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) + \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{kk} W_{lk}^l + (t - CT) \sum_{k=1}^{n} W_{kk} \Pi_{ik} - \theta \sum_{k=1}^{n} W_{kk} W_{lk}^l
\]

\[
\sum_{k=1}^{n} \Pi_{ik} \alpha_{k,0} - \theta \sum_{k=1}^{n} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) \leq \varepsilon \left( \sum_{k=1}^{n} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) + \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{kk} W_{lk}^l + (t - CT) \sum_{k=1}^{n} W_{kk} \Pi_{ik} \right)
\]

\[
(1 - \theta) \sum_{k=1}^{n} \Pi_{ik} \alpha_{k,0} - \theta \sum_{k=1}^{n} \Pi_{ik} \beta_{k,0} - \varepsilon \left( \sum_{k=1}^{n} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) + \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{kk} W_{lk}^l \right) \leq \varepsilon (t - CT) \sum_{k=1}^{n} W_{kk} \Pi_{ik}
\]

Assigning agent’s identity:

\[
(1 - \theta) \sum_{k=1}^{n} \Pi_{ik} \alpha_{k,0} - \theta \sum_{k=1}^{n} \Pi_{ik} \beta_{k,0} - \varepsilon \left( \sum_{k=1}^{n} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) + \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{kk} W_{lk}^l \right) + CT \leq t_i
\]

2. Case: \(|E[y_{i,t}] - y_\infty| = -|E[y_{i,t}] - y_\infty|\)

\[
(\theta - 1) \sum_{k=1}^{n} \Pi_{ik} \alpha_{k,0} + \theta \sum_{k=1}^{n} \Pi_{ik} \beta_{k,0} - \varepsilon \left( \sum_{k=1}^{n} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) + \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{kk} W_{lk}^l \right) + CT \leq t_i
\]

\[
\varepsilon \sum_{k=1}^{n} W_{kk} \Pi_{ik}
\]
Generalizing, we can write:

\[
\begin{align*}
\varepsilon \sum_{k=1}^{n} W_{kk} \Pi_{ik} & \quad + CT \leq t_i \\
\varepsilon \left( \sum_{k=1}^{n} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) + \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{kk} W_{lp} \right) & \quad - \varepsilon \sum_{k=1}^{n} W_{kk} \Pi_{ik} \\
\left(1 - \theta\right) \sum_{k=1}^{n} \Pi_{ik} \alpha_{k,0} - \theta \sum_{k=1}^{n} \Pi_{ik} \beta_{k,0} & \quad - \varepsilon \left( \sum_{k=1}^{n} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) + \sum_{l=0}^{CT-1} \sum_{k=1}^{n} W_{kk} W_{lp} \right) + CT \\
\end{align*}
\]

ii) In the same manner, we express the expected time for the society with stubborn agents:

\[
t_i \geq \frac{1}{\varepsilon} \varepsilon \left( \sum_{k \in B} \Pi_{ik} \alpha_{k,0} - \theta \sum_{k \in S \cup B} \Pi_{ik} \beta_{k,0} - \varepsilon \left( \sum_{k \in S \cup B} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) + \sum_{l=0}^{CT-1} \sum_{k \in N \setminus S} W_{kk} W_{lp} \right) + CT \right)
\]

iii) And we obtain the expected time for the society with polarized agents, taking into account \( y_{i,\infty} = \theta - \kappa \left( \sum_{k \in P^{-}} \Pi_{ik} - \sum_{k \in P^{+}} \Pi_{ik} \right) \)

\[
t_i \geq \frac{1}{\varepsilon} \left( \left(1 - y_{i,\infty}\right) \sum_{k \in P} \Pi_{ik} \alpha_{k,0} - y_{i,\infty} \sum_{k \in P} \Pi_{ik} \beta_{k,0} \right) + \kappa \left( \sum_{k \in P^{-}} \Pi_{ik} - \sum_{k \in P^{+}} \Pi_{ik} \right) \sum_{l=0}^{CT-1} \sum_{k \in N \setminus P} W_{kk} W_{lp} + 2 \left( \sum_{k \in P^{-}} \Pi_{ik} \sum_{l=0}^{CT-1} \sum_{k \in P^{+}} W_{lk} W_{lp} - \sum_{k \in P^{-}} \Pi_{ik} \sum_{l=0}^{CT-1} \sum_{k \in P^{+}} W_{lp} \right) - \varepsilon \left( \sum_{k \in P} \Pi_{ik} (\alpha_{k,0} + \beta_{k,0}) + \sum_{l=0}^{CT-1} \sum_{k \in P} W_{kk} W_{lp} \right) + CT
\]

\[\square\]

**References**


42